Math 4200
Fri 18 Nov.

HW for Wed 11/25:
8.2 1c, 2, 9
+ Exercise: Use Laplace transform to show "linear factors" partial fractions algorithm ~~always~~ works, even with repeated roots!

Multi-variable FTC $\sim$ a key tool in understanding PDE's.

1st recall 1-variable version:

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F'(x) = f(x)$$

or

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

A good way to think of this:

$$F'(x) \approx \frac{F(x+h) - F(x)}{h}$$

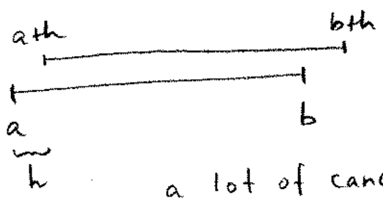
so

$$\int_a^b F'(x) \overset{1}{\approx} \int_a^b \frac{F(x+h)-F(x)}{h}\,dx$$

$$= \frac{1}{h}\int_a^b F(x+h)\,dx - \frac{1}{h}\int_a^b F(x)\,dx$$

$z = x+h$
$dz = dx$
(translation of variables)

$$= \frac{1}{h}\int_{a+h}^{b+h} F(z)\,dz - \frac{1}{h}\int_a^b F(x)\,dx$$



a lot of cancellation!

$$= \frac{1}{h}\int_b^{b+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx$$

$$\overset{2}{\approx} F(b) - F(a)$$

In fact, as $h \to 0$ you can deduce

$$\int_a^b F'(x) = F(b) - F(a)$$

[For first $\approx$ you need e.g. $F'(x)$ continuous on $[a,b]$
For second $\approx$ you need e.g. $F(x)$ continuous at a & b]

This is prep for §2.5
– understanding significance of Poisson integral formula for harmonic fcns
Then Fourier transform discussion of heat eqtn, partly in §4.3, mostly extra
Contour integrals appear in both topics.
Then, chapter 5.

Details:

1 $\approx$: By MVT

$$\frac{F(x+h)-F(x)}{h} = F'(c_x), \quad |x-c_x| < h$$

so

$$\left|\int_a^b F'(x) - \left(\frac{F(x+h)-F(x)}{h}\right)dx\right| \le \int_a^b |F'(x) - F'(c_x)|\,dx, \quad |x-c_x|<h$$

If F' unif cont on $[a,b]$ then $\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|x-c_x| < \delta \Rightarrow |F'(x) - F'(c_x)| < \varepsilon$

in which case,

$$\le \int_a^b \varepsilon\,dx = \varepsilon(b-a) \quad ■$$

2 $\approx$:

$$\left|\frac{1}{h}\int_b^{b+h} F(x)\,dx - F(b)\right| = \left|\frac{1}{h}\int_b^{b+h} F(x) - F(b)\,dx\right|$$

F cont at $b \Rightarrow \forall \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|F(x) - F(b)| < \varepsilon$ when $|x-b| < \delta$

in which case

$$\le \frac{1}{h}\int_b^{b+h} \varepsilon\,dx = \varepsilon \quad ■$$

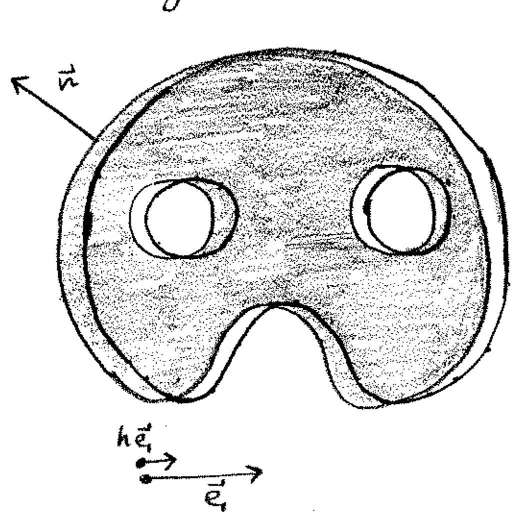
The same idea is behind $\mathbb{R}^m$ version
(although proof usually follows a technically simpler route)

Illustrate when $n=2$

Domain D (shaded), ∂D = bdry curve

$D_h := \{ z = x + h\vec{e}_1$ s.t. $x \in D \}$

is translation by amt h in $\vec{e}_1$ direction



$\vec{n}$ = unit exterior normal

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}$

$$\iint_D \frac{\partial F}{\partial x}\, dA \overset{①}{\approx} \iint_D \frac{F(\vec{x}+h\vec{e}_1) - F(\vec{x})}{h}\, dA$$

$$= \frac{1}{h}\iint_D F(\vec{x}+h\vec{e}_1)\, dA - \frac{1}{h}\iint_D F(\vec{x})\, dA$$

$\vec{z} = \vec{x} + h\vec{e}_1$

$dA_z = dA_x$

(translation of domains

$$= \frac{1}{h}\iint_{D_h} F(\vec{z})\, dA_z - \frac{1}{h}\iint_D F(\vec{x})\, dA_x$$

$\overset{①}{\approx}$ and $\overset{②}{\approx}$ are messy to prove, so rigorous proof is exposited slightly differently in general (page 4), but with this idea hiding behind it.

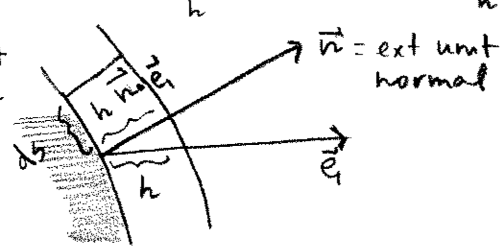
lots of cancellation in the middle

$S_h^+ = D_h \setminus D$ = unshaded strip(s)

$S_h^- = D \setminus D_h$ = shaded strip(s)

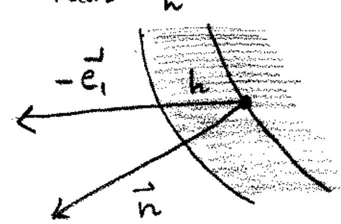
$$* \quad = \frac{1}{h}\iint_{S_h^+} F(x)\, dA - \frac{1}{h}\iint_{S_h^-} F(x)\, dA$$

near S_h^+



$dA \approx ds\, h(\vec{n}\cdot\vec{e}_1)$

near S_h^-



$dA \approx h\, ds\, (\vec{n}\cdot(-\vec{e}_1))$

So
$$* \overset{②}{\approx} \int_{\partial D} F\vec{e}_1\cdot\vec{n}\, ds$$

(h's cancel, also signs in 2nd term)

In the limit
$$\boxed{\iint_D \frac{\partial F}{\partial x}\, dA = \int_{\partial D} F\vec{e}_1\cdot\vec{n}\, ds}$$

$\mathbb{R}^1$	$\mathbb{R}^m$
$\int_a^b F'(x)\,dx = F(b)-F(a)$	$\int_{D_m} \frac{\partial F}{\partial x_i}\, dVol_m = \int_{\partial D_m} F(\vec{e}_i \cdot \vec{n})\, dVol_{m-1}$ (unit ext. normal)

SAME!

Special forms: $\boxed{\text{Divergence Theorem}}$

$\mathbb{R}^2$: $\operatorname{div}\begin{bmatrix} P \\ Q \end{bmatrix} := P_x + Q_y$

$$\iint_D P_x\, dA = \int_{\partial D} P\, \vec{e}_1 \cdot \vec{n}\, ds$$

$$+ \quad \iint_D Q_y\, dA = \int_{\partial D} Q\, \vec{e}_2 \cdot \vec{n}\, ds$$

$$\iint_D \operatorname{div}\begin{bmatrix} P \\ Q \end{bmatrix} dA = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \vec{n}\, ds$$

$\vec{F}(\vec{x})$ a vector field

in $\mathbb{R}^m$

$$\boxed{\int_{D_m} \operatorname{div}\vec{F}\, dVol_m = \int_{\partial D_m} \vec{F}\cdot\vec{n}\, dVol_{m-1}}$$

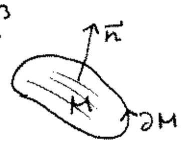
Green's & Stoke's Thm

Green: $\boxed{\iint_D Q_x - P_y\, dA = \oint_{\partial D} P\,dx + Q\,dy}$

(We used this to give initial proofs of Cauchy's Theorem and the Cauchy integral formula)

Stokes:

$M^2 \subset \mathbb{R}^3$ a surface in $\mathbb{R}^3$



$$\boxed{\iint_M (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dA = \oint_{\partial M} \vec{F}\cdot d\vec{x}} = \oint_{\partial M} P\,dx + Q\,dy + R\,dz$$

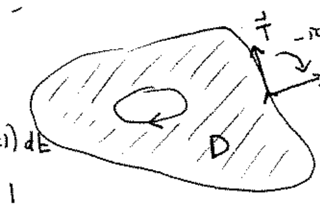
(Stoke's generalizes Green's from flat surfaces to general ones)
(Stokes follows from Greens as well.)

equivalent to $m=2$ div thm:

$$\iint_D Q_x - P_y\, dA = \iint_D \operatorname{div}(Q,-P)\, dA \overset{n=2 \text{ div}}{=} \int_{\partial D} (Q,-P)\cdot\vec{n}\, ds$$

$\gamma(t) = \begin{bmatrix} x(t) \\ y'(t) \end{bmatrix}$

$\gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ $\quad ds = \|\vec{\gamma}'(t)\|\, dt$



unit tangent $T = \frac{1}{\|\gamma'(t)\|}\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\int_a^b (Q(\gamma(t)), -P(\gamma(t)))\cdot(y'(t), -x'(t))\, dt$$

$$= \int_{\partial D} P\,dx + Q\,dy \;!$$

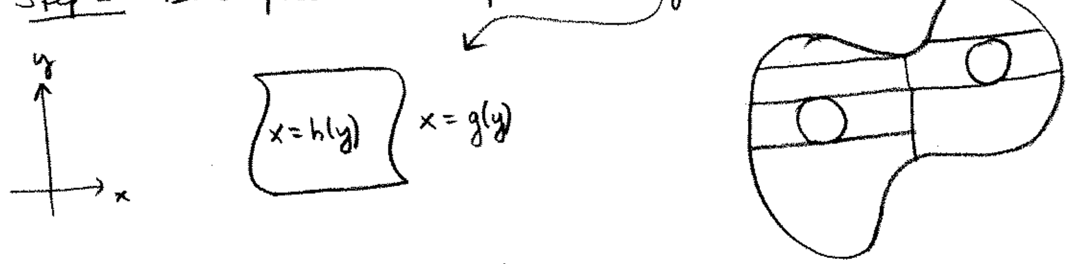
$\vec{n} = \operatorname{rot}_{-\pi/2} \vec{T} = \frac{1}{\|\gamma'(t)\|}\begin{bmatrix} y' \\ -x' \end{bmatrix}$

$\vec{n}\, ds = \begin{bmatrix} y' \\ -x' \end{bmatrix} dt$

The way towards a rigorous proof of m-variable FTC
(illustrated for $m=2$)

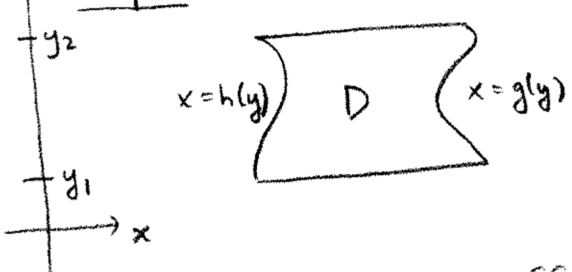
to show $\displaystyle\iint_D \frac{\partial F}{\partial x}\, dA = \int_{\partial D} F\,\vec{e}_1\cdot\vec{n}\, ds$

<u>Step 1</u> Decompose D into pieces of type



if can show for basic piece,
add over all pieces, notice interior edge contributions from adjacent pieces cancel, so get thm.

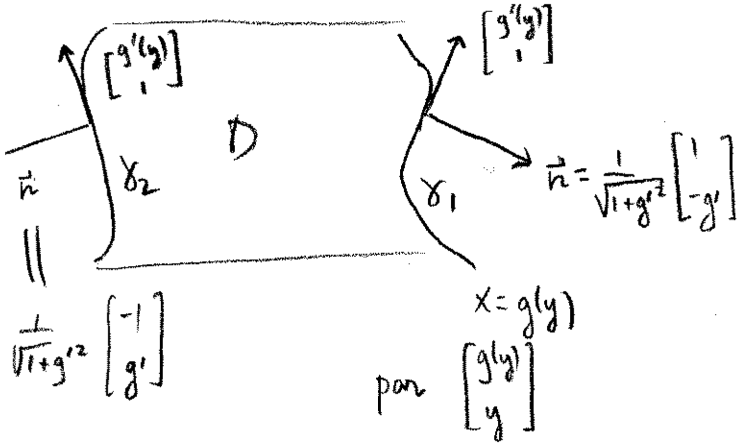
<u>Step 2</u>



Fubini

$$\iint_D \frac{\partial F}{\partial x}\, dA = \iint_D \frac{\partial F}{\partial x}\, dx\,dy = \int_{y_1}^{y_2}\left(\int_{h(y)}^{g(y)} \frac{\partial F}{\partial x}\, dx\right) dy$$

$$= \int_{y_1}^{y_2} F(y, g(y)) - F(y, h(y))\, dy \qquad \text{1-var FTC}$$



$$= \int_{\gamma_1} F\, dy - \int_{\gamma_2} F\, dy$$

but on γ_1, γ_2, $ds = \sqrt{1+g'^2}\, dy$; $dy = \frac{1}{\sqrt{1+g'(y)^2}}\, ds$

$$= \int_{\partial D} F(\vec{e}_1\cdot\vec{n})\, ds \quad \blacksquare$$

Sample PDE derivation (See Strauss notes for more - most derivations depend on some use of FTC.) (Also see page 349 text)

Heat (diffusion) equation

$u(\vec{x},t)$ can represent temperature at position $\vec{x}$, time t. Here $\vec{x} \in \Omega$, $t \in \mathbb{R}^+$
(or it can represent concentration of some solute)

Model: • Assume temp (or solute) diffuses from warmer to cooler (concentrated to less concentrated), moving in direction of greatest decrease, i.e. $-\nabla u$ (spatial gradient $= \nabla u$).

• no energy (or solute) sources or sinks inside the domain

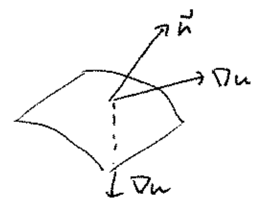
The precise way to encode this model, using units for u = temp = energy density (up to a constant) is, for any "test" subdomain $\mathcal{O} \subset \mathcal{U}$

$$E(t) := \iiint_{\mathcal{O}} c\,\rho\, u(\vec{x},t)\, dV_{\vec{x}}$$

total energy inside $\mathcal{O}$

c: specific heat (energy required to heat 1 mass unit by 1 temp unit)

ρ: density (mass/vol)



Model $$\frac{dE}{dt} = \iint_{\partial\mathcal{O}} \kappa\, \nabla\vec{u}\cdot\vec{n}\, dA_{\vec{x}}$$

κ: thermal conductivity

total energy in $\mathcal{O}$ only changes by energy flowing through $\partial\mathcal{O}$ in direction of $-\nabla u$.

Assuming u is differentiable enough, pass $\frac{d}{dt}$ through integral on left, apply div. thm on right

$$\Rightarrow \iiint_{\mathcal{O}} c\,\rho\, u_t(\vec{x},t)\, dV_{\vec{x}} = \iiint_{\mathcal{O}} \kappa\, \underbrace{\operatorname{div}(\nabla\vec{u})}_{=\Delta u}\, dV_x$$

$$\iiint_{\mathcal{O}} (c\rho u_t - \kappa\Delta u)\, dV_x = 0$$

This can be true for all test domains iff the integrand is identically zero (assuming it's continuous)

e.g. let $\mathcal{O} = B(\vec{x}_0, r) = \{\vec{x} \mid \|x - x_0\| < r\}$

$$\frac{1}{\mathrm{Vol}(B(\vec{x}_0,r))} \iiint_{B(\vec{x}_0,r)} f(\vec{x})\, dV \to f(\vec{x}_0)$$

as $r \to 0$ if f is cont at x_0

$$\therefore \boxed{u_t = \frac{\kappa}{c\rho}\Delta u \quad \text{Heat Eqtn} \qquad 0 = \Delta u \quad \text{Laplace}}$$

Laplace: if sol'n is stationary in time

Natural boundary value problems:

① $u(\vec{x}) = f(\vec{x})$ on $\partial\Omega$ (Dirichlet)
- temp on $\partial\Omega$ is fixed.

② $\nabla\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$ (Neumann)
- no flux on $\partial\Omega$, i.e. insulated bdry for heat, impermeable bdy for solutes.