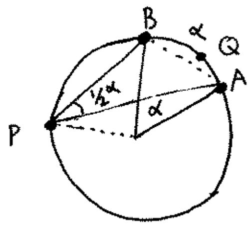
Math 4200
Wed 11/23

next HW batch due Fri 12/2

Class exercises: (I was Laplace problem last week)

$0<\alpha\le\pi$

II. Use trigonometry (mainly the fact that interior angles of Δ's add up to π) to show that on the unit circle, if we consider an arc of angle (& length) α, connecting A to B

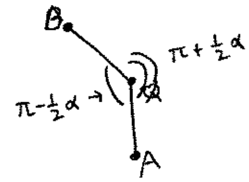


(a) and if P is any point on the unit circle not on the arc, then

$$\measuredangle APB = \tfrac{1}{2}\alpha$$

(b) if Q is any point on the arc, then

$$\measuredangle AQB = \pi - \tfrac{1}{2}\alpha$$



(c) Use (a), (b) to show that the solution to Laplace's eqtn $\Delta u=0$ in unit disk

with $\begin{cases} u(e^{i\theta}) = 1, & 0<\theta<\pi/2 \\ u(e^{i\theta})=0 & \frac{\pi}{2}<\theta<2\pi \end{cases}$

equals $\left[\operatorname{Im}\left(\log\left(\frac{z-i}{z-1}\right)\right) - \frac{\pi}{4}\right]\frac{1}{\pi}$

choose $0<\arg(z-i)<\pi$, $-\frac{\pi}{2}<\arg(z-1)<\pi/2$

(The general formula, $0\le\alpha\le\pi$, is

$$u(z) = \left(\arg\left(\frac{z-B}{z-A}\right) - \frac{\alpha}{2}\right)\frac{1}{\pi}$$)

[this was the $u(\rho e^{i\phi})$ graped in Maple notes using Poisson integral formula.]

III In PDE's (3150) or 2280 you may have solved the Dirichlet problem for a harmonic function by "separation of variables" in polar coords. Here's what you would end up with:

Let $f(\theta)$ represent the boundary values $u(e^{i\theta})$ for which you want to find a harmonic fcn $u(\rho e^{i\phi})$, $\rho<1$.

Let $f(\theta)$ have Fourier series

$$f \sim a_0 + \sum_{n=1}^{\infty} a_n\cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta \quad \text{(av. val.)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \quad n\ge 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \quad n\ge 1$$

"Fourier coeffs".

Then (with hand waving),

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} a_n \underbrace{r^n\cos n\theta}_{Re(z^n)} + \sum_{n=1}^{\infty} b_n \underbrace{r^n \sin n\theta}_{Im(z^n)}$$

was produced as the solution to $\begin{cases}\Delta u=0 & |z|<1 \\ u(e^{i\theta}) = f(e^{i\theta}) & \text{on } \partial D\end{cases}$

From pages 1-2 on Monday 11/21 notes, we produced the Poisson Integral formula:

geometric series, see page 1 11/21

$$u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(e^{i\theta})}_{f(\theta)\text{ in this case}} \left[\frac{1}{1-z/\zeta} + \frac{\bar z\zeta}{1-\bar z\zeta}\right] d\theta$$

$z=\rho e^{i\phi}$, $|z|<1$, $\zeta = e^{i\theta}$

$\boxed{\text{SHOW this produces the separation of variables soltn (except using } \rho,\phi \text{ rather than } r,\theta)}$