Math 4200
Final exam review sheet
December 7, 2005

The final exam is this Monday, December 12, here in JTB 120, from 10:30-12:30 (but I won't kick anyone out until 1:00). The exam is comprehensive. Precisely, the text material covered is chapters 1-5.2 and 8.1-8.2. On the exam you will be asked to pick six of ten problems, and it is possible that one or so of these problems could deal with the supplemental material we covered, related to PDE's. At least three of the exam problems will be chosen directly from our first two midterms (which are posted, along with solutions, on our home page), and at least two more will be favorites from homework. Several problems will ask you to prove key theorems from complex analysis.

We have our usual problem session tomorrow, for chapter 5 material. If there is interest we could also have a more general review session, in which you could ask questions from throughout the course, from old midterms, or from the final exam I gave last time I taught this course, for example. We could hold such a problem session on Friday morning of this week, if there is interest.

I think of basic complex analysis as following from (only) several key ideas, and the core of these ideas is what I hope you carry away from the course:

(0) Complex number algebra and geometry: addition, multiplication, conjugation, inverse, Euler's formula and the polar form of complex numbers. It is essential to understand this algebra and geometry, in order to understand the basic analytic functions.

(1) f(z)=f(x+iy)=u(x,y)+iv(x,y) is complex differentiable at z_0 (the limit definition of complex derivative) iff F(x,y)=[u(x,y),v(x,y)] is real differentiable, with derivative matrix a rotation dilation. This circle of ideas includes the Cauchy Riemann equations, the chain rule for $f(\gamma(t))$, where f is analytic and γ is a curve. It also includes the calculus of complex differentiation and the inverse function theorem for analytic functions. **I am certain to ask a question related to this circle of ideas.**

(2) Conformal transformations (bijections) between specified open, connected domains. (This ties into topic (1).) This includes the zoo of transformations we studied in chapter 1 (all the functions on your scientific calculator), including the concept of branches and branch cuts, along with the fractional linear transformations and RMT in chapter 5.1-5.2. **I am certain to ask a question related to this circle of ideas.**

(3) Contour integrals of analytic functions,

$$\int_\gamma f(z)\,dz\,.$$

how to compute by parameterization; the real and imaginary part are each real line integrals; why you expect such an integral to be zero when γ is the oriented boundary of a domain (Green's Theorem and CR equations!) This also yields intuitively reasonable explanations of the Cauchy integral formula and the residue theorem. **I am quite likely to ask a question related to this circle of ideas.**

(4) The circle of theorems related to the precise Cauchy integral formula:
(4a) f analytic in a disk ==> rectangle theorem in disk ==> antiderivative in disk.
local antiderivatives implies the deformation theorem for two homotopic closed curves in an domain A (which says that the contour integrals over the two curves must agree). In particular, for a simply connected domain, contour integrals of analytic functions over closed curves must be zero, implying

global antiderivatives exist.
(4b) definition of index and the derivation of the Cauchy integral formula, using (4a).
(4c) CIF for derivatives. Liouville. FTA. mean value and maximum modulus principle properties.**I am certain to ask at least one question related to this circle of ideas (especially 4a-4b), since the Cauchy integral formula is the basis for almost every interesting result in complex analysis.**

(5) Power series and Laurent series.
(5a) radius of convergence for power series (and uniform absolute convergence inside), and complementary result for power series in negative powers. Annulus of convergence consequence for a series with positive and negative powers. The fact that the resulting sums are analytic, using the fact that uniform limits of analytic functions are analytic (uses CIF!).
(5b) derivation and uniqueness of power series and Laurent series for analytic functions in disks and annuli. (The amazing uses of geometric series.)
(5c) uniqueness of analytic extension (cloning theorem), and its use in determining radius of convergence for analytic function power series.
(5d) finding power and Laurent series for given functions, e.g. method of equating coefficients...and you better know the basic power series if you hope to get started! **I am certain to ask at least one question related to this circle of ideas.**

(6) Residue calculus
(6a) computing residues
(6b) statement and proof of the residue theorem.
(6c) computing integrals with the residue theorem, or with some limiting process which starts with the residue theorem. See second midterm review notes for the specific integral types you should expect.
(6d) summing series with residue calculus.
I am certain to ask at least one question related to this circle of ideas, in particular there will be at least one type (6c) question!

(7) Applications to DE's and PDE's
(7a) harmonic functions:
harmonic conjugates in simply connected domains.
mean value and maximum modulus principle.
harmonic functions in the disk. Poisson integral solution to the Dirichlet problem in the disk. (If I ask about this I will provide formulas and maybe ask you to use them or explain facts related to them.)
harmonic composed with analytic is harmonic, lets you find harmonic functions in one domain by finding harmonic functions in another.
(7b) transforms: Laplace and Fourier. A Laplace question is possible, Fourier is unlikely.
It is likely I will ask a question related to this circle of ideas.

Math 4200
December 11, 2002
FINAL EXAM

Each complete problem below is worth 25 points. Choose any six out of the ten problems to do. Indicate clearly which six you want graded, if you have attempted more. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) Define what it means for a function f(z) to be complex differentiable (analytic) at a point z_o in **C**.

(7 points)

1b) State and prove a theorem which relates complex differentiability of f(x+iy)=u(x,y) + iv(x,y) at $z_o=x_o+i\,y_o$ to the Cauchy-Riemann equations and the real differentiability of

$$F(x,y)=\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

(18 points)

2) Find a branch of

$$f(z)=\sqrt{z^2-z}$$

in a suitable simply connected domain. Make sure to define f precisely.

(25 points)

3a) Find the Laurent series for the function

$$f(z)=\frac{1}{z^2+2\,z}$$

valid in the annulus $0<|z|<2$.

(10 points)

3b) Use your work from part (3a) to compute the contour integral (counterclockwise)

$$\int_{|z|=1} \frac{1}{z^2+2\,z}\,dz$$

(5 points)

3c) Check your work in (3b) using the residue theorem.

(10 points)

4a) Define what it means for a domain to be simply connected

(5 points)

4b) Prove that the complex plane with the origin deleted is not simply connected.

(20 points)

5) Use a contour integral to compute

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+4}\, dx$$

As always, justify all steps.

(25 points)

6) Use a contour integral to find

$$\int_{0}^{2\pi} \cos(\theta)^6 \, d\theta$$

(25 points)

7a) Consider the upper half disk consisting of points for which $|z|<2$ and $\mathrm{Re}(z)>0$. Find a conformal map f of this region to the upper half plane so that $f(i)=i$.

(20 points)

7b) Is your map in (7a) unique? Explain.

(5 points)

8a) State precisely what it means for two closed curves γ_0 and γ_1 to be homotopic in a domain A.

(8 points)

8b) Find a homotopy between the curves

$$\gamma_0(t)=e^{(i\,t)}+2\,e^{(6\,i\,t)}$$

and

$$\gamma_1(t)=e^{(6\,i\,t)}$$

in the complex plane with the origin deleted. (Here we take t between 0 and 2*Pi.) As always, justify!

(7 points)

8c) Use (8b) to find the value of

$$\int_{\gamma_0} 3\,\frac{1}{z}\,dz$$

(10 points)

9a) Prove that there is no conformal map from the entire complex plane to the unit disk.

(10 points)

9b) Prove that the only conformal bijections for which the entire complex plane is the domain, are the non-constant affine functions f(z)=az+b.

(15 points)

10) State and prove a theorem from this course which you find interesting, and which is important. (Of course, you should choose a theorem which you have not proven elsewhere on this exam.)

(25 points)