Math 4200
Exam 2 Solutions

1 a) Consider $G(\zeta) := \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$

Then $G(\zeta)$ is analytic in $A \setminus \{z\}$, and continuous at z
This is enough to prove the rectangle lemma (from chapter 3), even in a neighborhood of z, so that G has an antiderivative in any subdisk contained in A.
Thus the homotopy theorem from chapter 3 holds,

$$\int_{\gamma_1} G(\zeta)\,d\zeta = \int_{\gamma_2} G(\zeta)\,d\zeta$$

whenever γ_1, γ_2 are p.w. C' closed curves, homotopic as closed curves in A.

Since γ is homotopic (as closed curves), to a point in A, deduce

$$\underbrace{\int_\gamma G(\zeta)\,d\zeta}_{\displaystyle \int_\gamma \frac{f(\zeta)}{\zeta - z} - \frac{f(z)}{\zeta - z}\,d\zeta} = \int_{pt} G(\zeta)\,d\zeta = 0$$

Thus $\displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)\int_\gamma \frac{1}{\zeta - z}\,d\zeta = f(z)\,2\pi i\, I(\gamma; z)$ by definition of index.

1 b) Start with $\displaystyle f(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$

The right hand side can be differentiated with respect to z, arbitrarily often, and the derivatives can be passed thru the integral sign (this can be justified by proving the difference quotients converge uniformly to the z-deriv along γ). Thus

$$f^{(n)}(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \left[\left(\frac{d}{dz}\right)^n \frac{f(\zeta)}{\zeta - z}\right] d\zeta = \frac{1}{2\pi i}\, n! \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

1 c) $\displaystyle f'(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$

i.e. $\displaystyle f(z) - f(z_0) = \int_{\substack{\text{any path} \\ \text{from } z_0 \text{ to } z}} f'(\zeta)\,d\zeta = 0$!

If $|f(\zeta)| \le M \;\; \forall \zeta \in \mathbb{C}$

$$|f'(z)| \le \frac{1}{2\pi}\int_{|\zeta - z| = R} \left|\frac{f(\zeta)}{(\zeta - z)^2}\right| |d\zeta| \le \frac{1}{2\pi}\frac{M}{R^2}\, 2\pi R = \frac{M}{R}$$

let $R \to \infty \Rightarrow |f'(z)| = 0 \;\; \forall z \in \mathbb{C}$
$\Rightarrow f'(z) = 0 \Rightarrow f(z) = C.$ ∎

2a) $\int_\gamma f(z)\,dz = \oint_{|z-z_0|=r_1} \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}\,dz + \oint_{|z-z_0|=r_1} \sum_{n=0}^{\infty} a_n (z-z_0)^n\,dz$

$r < r_1 < R$

$= \sum_{n=1}^{\infty} \int_{|z-z_0|=r_1} \frac{b_n}{(z-z_0)^n}\,dz + \sum_{n=0}^{\infty} \int_{|z-z_0|=r_1} a_n (z-z_0)^n\,dz$

each of these integrals is zero by the FTC, $\int (z-z_0)^n\,dz = \frac{(z-z_0)^{n+1}}{n+1}$

we may interchange the summation and integral because each series converges uniformly on the curve $|z-z_0|=r_1$

when $n \neq 1$, these integrals are zero by FTC, since $\int \frac{1}{(z-z_0)^n}\,dz = \left(\frac{-1}{n-1}\right)\frac{1}{(z-z_0)^{n-1}}$ for $n \neq 1$

when $n=1$,

$\int_{|z-z_0|=r_1} \frac{b_1}{z-z_0}\,dz = 2\pi i\, b_1$, since e.g.

$z = z_0 + r_1 e^{it}$

$dz = r_1 i e^{it}$

$\int_\gamma \frac{b_1}{z-z_0}\,dz = \int_0^{2\pi} b_1 \frac{1}{r_1 e^{it}} r_1 i e^{it}\,dt$

$= b_1 i \int_0^{2\pi} 1\,dt = 2\pi i\, b_1.$

Thus $\boxed{b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz}$

2b). Consider $f(z)(z-z_0)^k$

$= \sum_{n=1}^{\infty} b_n \frac{(z-z_0)^k}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^{k+n}$

We discussed $k=0$ in 2a, to find b_1.

If $k \in \mathbb{N}$, repeat the discussion in 2a) to deduce

$b_{k+1} = \frac{1}{2\pi i}\int_\gamma f(z)(z-z_0)^k\,dz$, $\boxed{b_k = \frac{1}{2\pi i}\int_\gamma f(z)(z-z_0)^{k-1}\,dz}$

if $k < 0, k \in \mathbb{Z}$, deduce

$a_{(-k-1)} = \frac{1}{2\pi i}\int_\gamma f(z)(z-z_0)^k\,dz$

$-k-1 = \ell$

$k = -(\ell+1)$ i.e. $\boxed{a_\ell = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{\ell+1}}\,dz}$ $\ell = 0, 1, \ldots$

3 a) $f(z) = \frac{e^z}{z \sin z}$ has pole of order 2 at origin (since $\sin z = z\,\rho(z)$, $\rho(0) \neq 0$)

$$= \frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + a_1 z + \cdots$$

so $\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = \left(\frac{b_2}{z} + b_1 + a_0 z + a_1 z^2 + \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)$

equating coefficients of z^k:

z^0: $1 = b_2$

z^1: $1 = b_1$

z^2: $\frac{1}{2} = -\frac{b_2}{6} + a_0$; so $a_0 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

z^3: $\frac{1}{6} = a_1 - \frac{b_1}{6}$; so $a_1 = \frac{1}{3}$

$$\boxed{f(z) = \frac{1}{z^2} + \frac{1}{z} + \frac{2}{3} + \frac{1}{3} z + \cdots}$$

3b) $\sin z = 0$ at integer multiples of π (only), so the Laurent series for f converges in the punctured disk $0 < |z| < \pi$
(and in no larger disk since $|f(z)| \to \infty$ as $z \to \pm\pi$, which would violate uniqueness of analytic extension if the outer radius of convergence was $> \pi$)

3 c) $\displaystyle\int_{|z|=4} \frac{e^z}{z \sin z}\, dz = 2\pi i \left(\underbrace{Res(f,0)}_{=1 \text{ from work above}} + \underbrace{Res(f,\pi)}_{\text{simple pole}} + Res(f,-\pi)\right)$

if $f = \frac{g}{h}$ $g(z_0) \neq 0$, $h(z_0)$ zero of order 1

$$res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

take $g(z) = e^z/z$

$h(z) = z \sin z$, $h'(z) = (\cos z) z + \sin z$

$\Rightarrow Res(f,\pi) = \frac{e^\pi}{\pi} \frac{1}{-1}$

$Res(f,-\pi) = \frac{e^{-\pi}}{(-\pi)(-1)}$

$$= 2\pi i \left(1 - \frac{e^\pi}{\pi} + \frac{e^{-\pi}}{\pi}\right)$$

4. $\displaystyle\int_0^\pi (\sin\theta)^4\, d\theta = \frac{1}{2} \int_{-\pi}^{\pi} (\sin\theta)^4\, d\theta = Int$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$Int = \frac{1}{2} \int_{|z|=1} \left[\left(\frac{1}{2i}\right)\left(z - \frac{1}{z}\right)\right]^4 \frac{dz}{iz} = \frac{1}{2}\, 2\pi i\, \frac{1}{i}\, Res\left(\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^4 \frac{1}{z}, 0\right)$$

$$= \pi \frac{1}{16} Res\left(\underbrace{\left(z - \frac{1}{z}\right)^4}_{} \frac{1}{z}, 0\right)$$

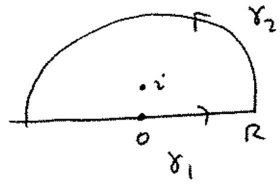
$$\frac{1}{z}\left(z^4 - 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) - 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}\right)$$

$$= \frac{\pi}{16} \cdot 6 = \boxed{\frac{3\pi}{8}}$$

5) $$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2} I$$

Let γ:

$\gamma = \gamma_1 + \gamma_2$



Let $f(z) = \dfrac{1}{(z^2+1)^2}$

$= \dfrac{1}{(z-i)^2(z+i)^2}$

$$I = \lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_{\gamma_2} |f(z)|\,|dz|$$

$$\le \int_{\gamma_2} \frac{1}{(R^2-1)^2}\,|dz| \qquad R>1 \text{ (reverse } \Delta \text{ ineq)}$$

$$= \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } R\to\infty.$$

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = 2\pi i\,(\mathrm{Res}(f(z), i))$$

$$(z-i)^2 f(z) = \frac{1}{(z+i)^2} = \phi(z)$$

$$= \phi(i) + \phi'(i)(z-i) + \cdots$$

near $z = i$.

$$\mathrm{Res}(f(z), i) = \phi'(i)$$

$$\phi'(z) = \frac{-2}{(z+i)^3}; \qquad \phi'(i) = \frac{-2}{(2i)^3} = -\frac{1}{4}\frac{1}{i^3} = \frac{1}{4i} =$$

$$\text{so } \int_{\gamma_1+\gamma_2} f(z)\,dz = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$\int_{\gamma_1} f(z)\,dz \to \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx, \qquad \int_{\gamma_2} f(z)\,dz \xrightarrow{R\to\infty} 0$$

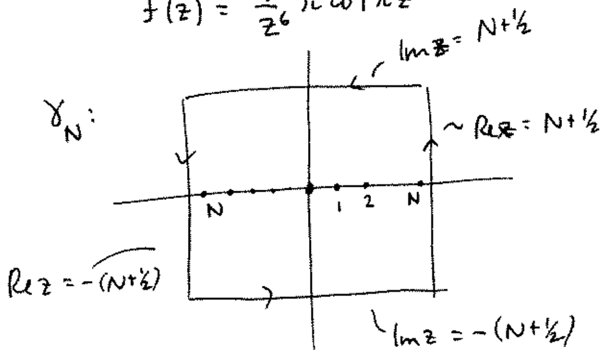
Therefore, $$\boxed{\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}}$$

6) $z\pi\cot\pi z = 1 - \frac{\pi^2}{3}z^2 - \frac{\pi^4}{45}z^4 - \frac{2\pi^6}{945}z^6 - \dots$

Consider

$f(z) = \frac{1}{z^6}\pi\cot\pi z$

γ_N:



f has poles at every integer.

for $n \neq 0$ an integer the pole is simple, and the residue is

$$\frac{g(n)}{h'(n)} \qquad g(z) = \frac{1}{z^6}\pi\cos\pi z, \quad h(z) = \sin\pi z, \quad h'(z) = \pi\cos\pi z$$

$$= \frac{1}{n^6}\frac{\pi\cos\pi n}{\pi\cos\pi n} = \frac{1}{n^6}$$

for $n=0$, the residue is the $\frac{1}{z}$ coeff of $\frac{1}{z^7}(z\pi\cot\pi z) = -\frac{2\pi^6}{945}$

$$\Rightarrow \int_{\gamma_N} \frac{1}{z^6}\pi\cot\pi z\,dz = 2\pi i\left(\sum_{\substack{n=-N \\ n\neq 0}}^{N}\frac{1}{n^6} - \frac{2\pi^6}{945}\right)$$

$$\left|\int_{\gamma_n} f(z)dz\right| \leq \int_{\gamma_n} |f(z)||dz| \leq \frac{1}{(N+\frac{1}{2})^6}\underset{\substack{\uparrow \\ \text{given bound for } |\cot\pi z| \text{ on } \gamma_N}}{\pi\cdot 2}\cdot 4(N+1) \to 0 \text{ as } N\to\infty.$$

Deduce $2\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{2\pi^6}{945}$, so $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}}$