Math 4200
Exam 1 Sol'ns

1. a) f is complex diff'ble at $z_0 \in \mathbb{C}$ iff $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} := f'(z_0)$ exists

b) if $f(x+iy) = u(x,y) + iv(x,y)$
then CR eqtns are: $u_x = v_y$
$u_y = -v_x$

f complex diff'ble at $z_0 = x_0+iy_0$ iff $F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}$ is real diff'ble at $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ & CR hold there

c) $f(z) = e^z = e^x e^{iy} = e^x\cos y + i e^x \sin y$

$u(x,y) = e^x\cos y$
$v(x,y) = e^x \sin y$ $\Rightarrow$ $u_x = e^x\cos y = v_y$
$u_y = -e^x\sin y = -v_x$

so CR hold everywhere
since u,v are C^1 (even C^∞)
deduce by (b) that $f = u+iv$ is analytic.

$f'(z) = f_x = -if_y$ (CR)
$= u_x + iv_x$
$= u+iv = f$, so $(e^z)' = e^z$

2 a) $\log z = \ln|z| + i\arg z$

b) $e^{\log z} = e^{\ln|z|}e^{i\arg z} = |z|e^{i\theta}$, $\theta = \arg z$
$= z$

so $\log z$ is local inverse fcn to exp (so is analytic)

c) $\frac{d}{dz} e^{\log z} = \frac{d}{dz} z = 1$
$e^{\log z}(\log z)' = 1$
$z(\log z)' = 1$
$(\log z)' = 1/z$ ∎

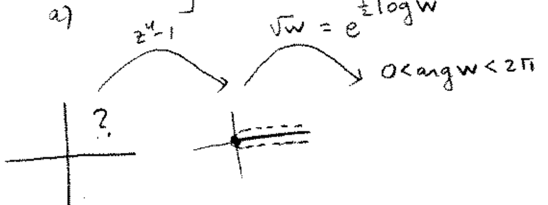
d) write $z = re^{i\theta}$ $-\pi<\theta<\pi$
$\Rightarrow z^2 = r^2e^{2i\theta}$

$\log z = \ln r + i\theta$
$2\log z = 2\ln r + i(2\theta)$
whereas $\log z^2 = \ln r^2 + i\underbrace{\arg(e^{2i\theta})}_{\text{value btwn } -\pi..\pi}$
$= 2\ln r$

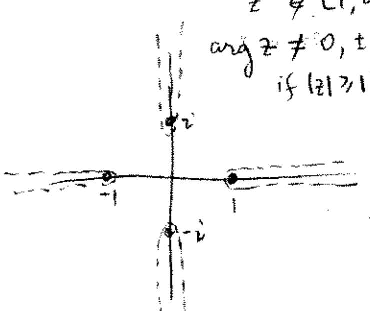
so if $-\pi < 2\theta < \pi$ then $\log z^2 = 2\log z$
$\underline{\underline{-\pi/2 < \theta < \pi/2}}$

if $\pi/2 < \theta < \pi$, then $\arg e^{2i\theta} = 2\theta - 2\pi$
if $-\pi < \theta < -\pi/2$ then $\arg e^{2i\theta} = 2\theta + 2\pi$.

3. one way:
a)



need $z^4-1 \notin [0,\infty)$
$z^4 \notin [1,\infty)$
$\arg z \neq 0, \pm\pi/2, \pi$
if $|z|\geq 1$, leads to domain



this domain is starshaped wrt 0,
so is simply connected.

so
$\sqrt{z^4-1} = e^{\frac12\log(z^4-1)}$, $0<\arg(z^4-1)<2\pi$

You could also do this by writing
$\sqrt{z^4-1} = \sqrt{z-1}\sqrt{z+1}\sqrt{z-i}\sqrt{z+i}$
and pick branches for each of the 4 $\sqrt{\ }$'s on RHS

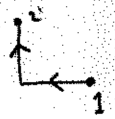
3 b) $f'(z) = e^{\frac{1}{2}\log(z^4-1)} \frac{1}{2}\frac{1}{z^4-1}(4z^3)$

$$= \frac{1}{\sqrt{z^4-1}} \frac{2z^3}{\cancel{z^4}}$$

3c) Since domain is simply connected contour integrals are path independent, and $F(z) := \int_0^z f(w)\,dw$ is an antideriv.

(γ any curve from 0 to z, in this case could take line segment)

4. a) $\int_\gamma 3z^2\,dz = z^3\Big]_1^i = i^3 - 1 = -i-1$

b)  $\int_\gamma 3z^2\,dz = \int_1^0 3x^2\,dx + \int_0^1 \underbrace{3(iy)^2}_{-3iy^2} i\,dy$

$z = x$, $dz = dx$ $\quad$ $z = iy$, $dz = i\,dy$

$$= x^3\Big]_1^0 - iy^3\Big]_0^1$$

$$= 0 - 1 + (-i - 0) = -1 - i \checkmark$$

4c) Let $F'(z) = f(z)$ on an open domain A containing the range of C^1 $\gamma: [a,b] \to \mathbb{C}$.

Then $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$

pf $\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt$

$$= \int_a^b \underbrace{F'(\gamma(t))\gamma'(t)}\,dt$$

$= \frac{d}{dt}F(\gamma(t))$ by the chain rule for curves

$= F(\gamma(t))\Big]_a^b$ by 1210 FTC on real & im parts of F.

$= F(\gamma(b)) - F(\gamma(a))$ ∎

5. $\gamma(t) = 2e^{it} + e^{6it}$ is homotopic in $\mathbb{C}\setminus\{0\}$ to $\gamma_1(t) = 2e^{it}$:

5a) $H(s,t) = 2e^{it} + (1-s)e^{6it}$ $\quad 0 \le s \le 1$

notice $|H(s,t)| \ge 2 - (1-s) \ge 1$ so image does stay in $\mathbb{C}\setminus\{0\}$.

5b) $\Rightarrow \int_\gamma \frac{1}{z}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz = \boxed{2\pi i}$

$z = 2e^{it}$, $dz = 2ie^{it}\,dt$

$\int_0^{2\pi} \frac{1}{2e^{it}} 2ie^{it}\,dt = 2\pi i \checkmark$

deformation thm: If γ_1 homotopic to γ_2 as closed curves in open A, and f analytic in A, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$

reverse Δ ineq: $|z^3 - 1| \ge |z^3| - 1 = R^3 - 1$

so $\frac{1}{|z^3-1|} \le \frac{1}{R^3-1}$

6. for $R > 1$,

a) $\left| \int_{|z|=R} \frac{1}{z^3-1}\,dz \right| \le \int_{|z|=R} \frac{1}{|z^3-1|}|dz| \le \int_{|z|=R} \frac{1}{R^3-1}|dz|$

$$= \frac{2\pi R}{R^3-1} \to 0 \text{ as } R \to \infty.$$

b) $\frac{1}{z^3-1}$ is analytic in $\mathbb{C}\setminus\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$

so for $R > 1$ all $\int_{|z|=R} \frac{1}{z^3-1}\,dz$ are equal, by the deformation thm.

Hence $\int_{|z|=2} \frac{1}{z^3-1}\,dz = \int_{\substack{|z|=R \\ R>1}} \frac{1}{z^3-1}\,dz \quad \forall R$. Let $R \to \infty \Rightarrow \int_{|z|=2} \frac{1}{z^3-1}\,dz = 0$.

You could also do this by partial fractions, but it would be a little tedious

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Exam 1 distribution