EXAM 1
Math 4200-1 October 3, 2005

Each complete problem below is worth 25 points. Choose any four out of the six problems to do. If you try 5 or 6 problems, indicate clearly which four you want graded. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) Define what it means for a function $f(z)$ to be complex differentiable at a point $z_0$ in $\mathbb{C}$. (7 points)

1b) What are the Cauchy-Riemann equations? Explain precisely how they are related to complex differentiability (proofs not required). (8 points)

1c) Prove that the function $f(z) = e^z$ is complex differentiable for all $z$ in $\mathbb{C}$, and identify its derivative. You may want to use your work in part (1b) rather than a direct argument. (10 points)

2a) Define $\log(z)$. (5 points)

2b) Prove $\exp(\log(z)) = z$ no matter what branch of the argument is used for the logarithm. (5 points)

2c) Use (2b) and the chain rule to deduce that $\log(z)$ is an antiderivative of $1/z$. (5 points)

2d) If you choose the principal branch of the logarithm, so that $-\pi < \arg(z) < \pi$, then explain for which $z$ the following identity does hold, and for which $z$ it doesn’t:

$$\log z^2 = 2 \log z$$

(10 points)

3a) Define a branch of the function $f(z) = \sqrt{z^4 - 1}$ on a suitable simply connected domain. (15 points)

3b) What is the derivative of $f(z)$? (5 points)

3c) Write down an expression which yields the antiderivative function for $f$, on your simply connected domain. Justify your answer. (Hint: This question is asking if you remember how to antidifferentiate an arbitrary analytic function on a simply connected domain.) (5 points)

4a) Let $\gamma$ be any piecewise $C^1$ curve from the point 1 to the point $i$ in the complex plane. Using the fundamental theorem of Calculus for contour integration, deduce the value of

$$\int_{\gamma} 3z^2\,dz$$

(8 points)

b) Compute the contour integral above by using the piecewise $C^1$ curve which goes along the x-axis from 1 to 0, and then up the imaginary axis from 0 to $i$. You should get the same answer as in part 4a. (10 points)

c) Prove the fundamental theorem of Calculus, for contour integrals of analytic functions over $C^1$ curves. (7 points)
5) Let 
\[ \gamma(t) = 2e^{(it)} + e^{(6it)} \]
as \( t \) ranges from 0 to \( 2\pi \). In case you’re having trouble visualizing the image curve, here is a picture of it:

5a) Find a homotopy from \( \gamma(t) \) to the circle \( \alpha(t) = 2e^{(it)} \), in the complex plane complement the origin. Verify that your homotopy avoids the origin. (12 points)

5b) State and use the deformation theorem to deduce the value of
\[ \int_{\gamma} \frac{1}{z} \, dz \] (13 points)

6a) Prove via estimates that for the closed curves \( \gamma \) which traverse the radius-\( R \) once circle counterclockwise,
\[ \lim_{R \to \infty} \int_{|z|=R} \frac{1}{z^3 - 1} \, dz = 0 \] (13 points)

6b) Using (6a) or some other method, deduce the value of the
\[ \int_{|z|=2} \frac{1}{z^3 - 1} \, dz \] (12 points)