EXAM 1

Math 4200-1 October 3, 2005

Each complete problem below is worth 25 points. Choose any four out of the six problems to do. If you try 5 or 6 problems, indicate clearly which four you want graded. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) Define what it means for a function f(z) to be complex differentiable at a point z_o in **C.**

(7 points)

1b) What are the Cauchy-Riemann equations? Explain precisely how they are related to complex differentiability (proofs not required).

(8 points)

1c) Prove that the function $f(z) = \mathbf{e}^z$ is complex differentiable for all z in C, and identify its derivative. You may want to use your work in part (1b) rather than a direct argument.

(10 points)

2a) Define log(z).

(5 points)

2b) Prove exp(log(z))=z no matter what branch of the argument is used for the logarithm.

(5 points)

2c) Use (2b) and the chain rule to deduce that log(z) is an antiderivative of 1/z.

(5 points)

2d) If you choose the principal branch of the logarithm, so that $-\pi < \arg(z) < \pi$, then explain for which z the following identity does hold, and for which z it doesn't:

$$\log z^2 = 2 \log z$$

(10 points)

3a) Define a branch of the function

$$f(z) = \sqrt{z^4 - 1}$$

on a suitable simply connected domain.

(15 points)

3b) What is the derivative of f(z)?

(5 points)

3c) Write down an expression which yields the antiderivative function for f, on your simply connected domain. Justify your answer. (Hint: This question is asking if you remember how to antidifferentiate an arbitrary analytic function on a simply connected domain.)

(5 points)

4a) Let γ be any piecewise C^1 curve from the point 1 to the point i in the complex plane. Using the fundamental theorem of Calculus for contour integration, deduce the value of

$$\int_\gamma 3\, z^2\, dz$$

(8 points)

b) Compute the contour integral above by using the piecewise C^1 curve which goes along the x-axis from 1 to 0, and then up the imaginary axis from 0 to i. You should get the same answer as in part 4a.

(10 points)

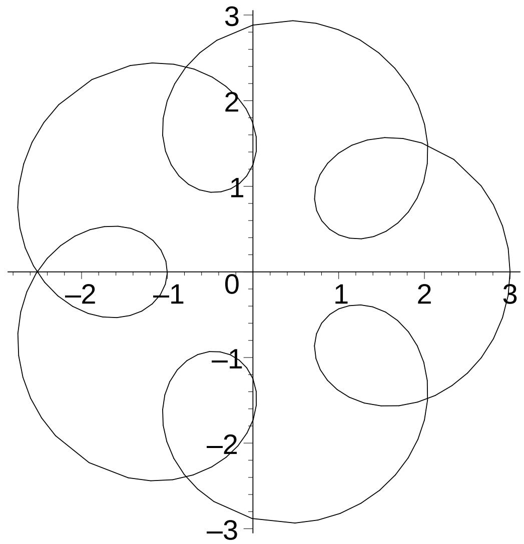
c) Prove the fundamental theorem of Calculus, for contour integrals of analytic functions over C^1 curves.

(7 points)

5) Let

$$\gamma(t) = 2\,\mathbf{e}^{(i\,t)} + \mathbf{e}^{(6\,i\,t)}$$

as t ranges from 0 to 2 π. In case you're having trouble visualizing the image curve, here is a picture of it:



5a) Find a homotopy from $\gamma(t)$ to the circle $\alpha(t) = 2\,\mathbf{e}^{(i\,t)}$, in the complex plane complement the origin. Verify that your homotopy avoids the origin.

(12 points)

5b) State and use the deformation theorem to deduce the value of

$$\int_{\gamma} \frac{1}{z}\,dz$$

(13 points)

6a) Prove via estimates that for the closed curves γ which traverse the radius-R once circle counterclockwise,

$$\lim_{R \to \infty} \int_{|z|=R} \frac{1}{z^3 - 1}\,dz = 0$$

(13 points)

6b) Using (6a) or some other method, deduce the value of the

$$\int_{|z|=2} \frac{1}{z^3 - 1}\,dz$$

(12 points)