Math 4200
Monday 12/5

Conformal transformations, cont'd

- In page 4 example, I should have checked $T_1(-1)$, as in notes.

- Focus on the LFT's we chose which transform upper half plane to unit disk (or reverse), as example for using one domain to solve Dirichlet Prob. for Laplace eqtn in another: (9.5.3)

$$f(z) = \frac{2 - z}{2 + z},$$

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

adjoint: $$\begin{bmatrix} i & 1 \\ 1 & 1 \end{bmatrix}$$

$$g(w) = \frac{2w + 1}{-w + 1} = \frac{i(1 - 1)}{-w + 1} = \frac{1}{i} \frac{(w + 1)}{w - 1}$$

Example (see HW II last week). Find harmonic func $u$ in disk, $u(\zeta) = 0 < \zeta < \pi$,

$$u = 1 \text{ for } e^{i\theta}, \quad 0 < \zeta < \frac{\pi}{2}$$

Consider

$$u(z) = \arg \left( \frac{2 - z}{2 + z} \right) = \arg \left( \frac{2 + z}{2 - z} \right)$$

Then $u(g(w)) = u \left( \frac{-i(w + i)}{w - 1} \right)$ solves the problem in disk, because

$$\text{harmonic (analytic)} = \text{harmonic (removable sing)}$$.

Note, $\arg(w) = \frac{1}{\pi} \arg \left( \frac{w - i}{w + i} \right)$

$$= \frac{1}{\pi} \arg \left( \frac{w - i + w - 1}{w - 1} \right)$$

$$= \frac{1}{\pi} \arg \left( \frac{(w - i)(1 - i)}{w - 1} \right) = \frac{1}{\pi} \left( \arg \left( \frac{w - i}{w + 1} \right) + \arg(1 - i) \right)$$

Isothems are circular arcs by LFT property, so are heat flux curves!
• The Möbius transformations of the unit disk: (also page 4, Ex.)

\[ f(z) = e^{i\theta} \left( \frac{2-z_0}{1-z_0 \overline{z}} \right) \]

Let \( T : D \to D \)

\[ z_0 \in D \text{ s.t. } T(z_0) = 0. \]

Let \( \arg(T'(z_0)) = \theta_0 \)

Consider \( f(z) = e^{i\theta} \left( \frac{2-z}{1-z} \right) \)

where \( \theta_0 \) is chosen so that \( \arg(f'(z_0)) = \theta \), too.

Then \( f^{-1} \circ T : D \to D \)

\[ z_0 \to z_0 \]

\[ \arg(f^{-1}(T'(z_0))) = 0 \Rightarrow f^{-1} \circ T = \text{id} \text{ by uniqueness in RMT} \Rightarrow T = f. \]

Note, if \( z_0 = z \)

then \( f^{-1}(z_0) = \left( \frac{2-z}{1-z z_0} \right) \)

So \( f \) maps unit circle to unit circle \& \( z_0 \) to 0.

Thus Disk to Disk 1 (see last page 8 notes for general discussion, or give special proof in this case).

• \( \text{SL}(2, \mathbb{R}) \) acting on upper half plane.

Consider \( \mathcal{H} = \{ (x, y) \mid y > 0 \} \).

\[ f(z) = \frac{z+c}{z+d} \]

\[ f^{-1}(w) = g(w) \]

\[ M(w) = e^{i\theta} \left( \frac{w-w_0}{1-w \overline{w_0}} \right) \]

every conformal bijection of \( \mathcal{H} \) is a composition LFT

\[ f \circ M \circ f^{-1} \]

(If \( T : H \to H \), then \( f \circ T \circ f^{-1} = M \) by proposition).

\[ T(z) = \frac{az + b}{cz + d} \]

where \( a, b, c, d \in \mathbb{R} \)

\[ \det = ad - bc = 1 \]

\[ T(i) = \frac{ai + b}{ci + d} \]

\[ = \frac{ai + b}{ci + d} \]

\[ = i \text{ in } \mathcal{H} \]

\[ T(0) = \frac{a+b}{d} \]

\[ T(\infty) = \frac{a}{c} \]

There's a nice (and fundamental) story in geometry, having to do with these transformations...
In Riemannian Geometry one studies manifolds which have (infinitesimal) elements of arc-length, so that one can study lengths of curves. (which leads to a lot more)

An isometry preserves $T: M \to M$ preserves all curve lengths.

\[ ds = \sqrt{dx^2 + dy^2} \] (so $||\dot{\gamma}(t)|| = \sqrt{\dot{x}^2 + \dot{y}^2}$ dt).

**Theorem:** The only orientation preserving isometries of $\mathbb{R}^2$ are

\[
T(x) = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} x + b = T_{\theta,b}(x)
\]

"proof": let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be any isometry (which preserves orientation)

Since $T$ preserves all curve lengths, by looking at short curve we deduce it preserves arc-length elements, so the derivative matrix $T$ at any point must be a rotation matrix (but, a priori, this rotation could change as you change pts)

Compose $T$ with some $T_{\theta,b}$ so that

\[
(T \circ T_{\theta,b})(0) = 0 \quad \text{and} \quad (T \circ T_{\theta,b})'(0) = I
\]

$T \circ T_{\theta,b}$ is isom which fixes origin & tangent vector at origin.

shortest lines $\to$ shortest lines

bottom pts $\to$ bottom pts.

$\Rightarrow$ pos $x$-axis $\to$ a curve which minimizes dist btm pts as it, with $O \to O$

$\Rightarrow$ pos $x$-axis $\to$ pos $x$-axis, as identity

$\Rightarrow$ each ray from origin $\to$ itself as identity.

$\Rightarrow$ $T \circ T_{\theta,b} = id$

$\Rightarrow$ $T = T_{\theta,b}^{-1}$

\[ \mathbb{S}^2 \subset \mathbb{R}^3 \]

\[
\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ s.t. \ x^2 + y^2 + z^2 = 1 \}
\]

by a completely analogous argument, one can show that the only orient. pres. isom of $\mathbb{S}^2$ are the $3 \times 3$ rotation matrices (called S.O.(3))

(i.e. compose with such a rot. matrix, to fix north pole & directions at north pole.)

By the way, using a stereographic chart for $\mathbb{S}^2$,

\[
\begin{align*}
\text{ds}^2 & = \frac{2}{(1 + x^2 + y^2)^2} (dx^2 + dy^2) \\
\end{align*}
\]
There is one more model space!!

$H^2 = \text{upper half space} \quad (\text{Hyperbolic space})$

$$ds = \frac{1}{y} \sqrt{dx^2 + dy^2}$$

and the isometries are

$$T(z) = \frac{az+b}{cz+d} \quad \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in SL(2, \mathbb{R})$$

generated by

$$T_1(z) = \lambda z, \quad \lambda > 0 \quad \left( \begin{array}{cc} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{array} \right)$$

$$T_2(z) = z + b \quad b \in \mathbb{R} \quad \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right)$$

$$T_3(z) = -\frac{1}{z} \quad \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$$

"straight forward to show" $T_1, T_2, T_3$ are isoms.

every orientation preserving isom must be conformal $\Rightarrow$ These are all.

$$\chi \cdot g (T_1 \circ T_2)'(t) = \lambda \chi'(t)$$

if $\chi(t) = x_0 + iy_0$

$$\|\chi'(t)\|_H^2 = \|\chi'(t)\|_E^2 \quad \|T_1 \circ T_2)'(t)\|_H^2 = \|\lambda \chi'(t)\|_E^2 = \|\chi'(t)\|_H^2 \frac{\lambda^2}{y_0^2}$$

Then, since $A_i$ "clearly" minimizes length let $\tilde{A}_i$:

all its image under isoms are also "geodesics"

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another model (chart)

for this space is the Poincare's disk $D = \{ z | |z| < 1 \}$

$$ds = \frac{2}{1-|z|^2} \sqrt{dx^2 + dy^2}$$

(Check to do this in sphere chart!)

and the page 1 $f(z), g(w)$

are isometries between $H^2 \& D^2$.

(the isometries of D are the Mobius transformations)
A theorem to justify our carelessness in conformal map checking:

**Theorem.** Let $A, S^2 \setminus \overline{A}$ be connected \((A \text{ open})\)

$\overline{B}, S^2 \setminus \overline{B}$ connected \((\overline{B} \text{ closed})\)

$f : A \to S^2$ conformal

$f : \overline{A} \to S^2$ continuous

$f(\partial A) \subseteq \overline{B}$

$\exists z_0 \in A \text{ s.t. } f(z_0) \in \overline{B}$

Then $f(A) = \overline{B}$, so if $f$ is 1-1 then $A$ and $B$ are conformally equivalent.

**Remark:** So, roughly speaking, conformal maps can't leak over their edges.

**Proof:** By pre and post-composing $f$ with inversions at $z \in \partial A$, $w \in \partial B$ we may reduce to the case $A \subseteq \Delta, B \subseteq \Delta$ bounded subsets.

So this is our picture:

1. $f(A)$ is open; this was a (He) problem in chapter 3.
   Idea: Let $z \in A$, $f(z) \neq 0 \Rightarrow$ local inverse $f^{-1} \Rightarrow \exists W(f(z), \epsilon) \subseteq f(A)$.

2. $B \subseteq f(A)$: pick $w \in B$, $\gamma : [a, b] \to A$
   $\gamma(a) = w_0$
   $\gamma(b) = w$
   Then $f(A) \cap \gamma = f(\gamma) \cap f(A)$, (since $f(A) \cap B = \emptyset$)

   So this set is open and closed in $\gamma$.

   i.e. $\{ t \in [a, b] \mid \gamma(t) \in f(A) \}$ is open and closed.

   Since $A \subseteq \Delta$ this set is entire $[a, b]$

   $W = \gamma(b) \in f(A)$

3. $f(A) \subseteq B$: Suffices to show $f(A) \subseteq B$

   since $f(A)$ is open.

   Let $w_1, \ldots, w_k \in \partial B$.

   $\max \{ |f(z)| \mid z \in A \}$. Since $w \not\in f(A)$

   $\gamma$ a path in $S^2 \setminus \overline{B}$ connecting $w_1, \ldots, w_k$.

   Then $f(A) \cap \gamma = f(\gamma) \cap f(A)$ is open and closed.

   subset of $\gamma$.

   Since $w \not\in f(A)$ set

   $f(A) \cap \gamma$ is empty $\Rightarrow w \not\in f(A)$

4. Hence $f(A) = B$.

   i.e. $f(A) \cup f(\partial A) = B \cup B$.

   $\Rightarrow f(\partial A) = \partial B$. 

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