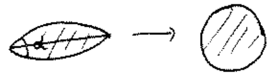
Math 4200

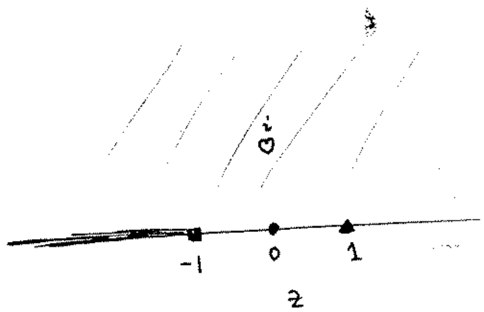
Monday 12/5

Conformal transformations, cont'd

- In page 4 example  Friday

  I should have checked $T_1'(-1)$, as in notes.

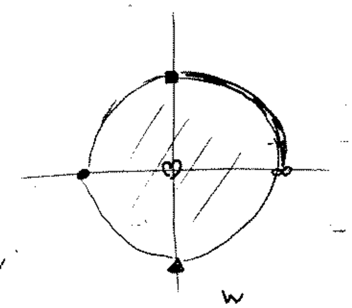
- Focus on "the" LFT's we chose which transform upper half plane to unit disk (& reverse), as example for using one domain to solve Dirichlet Prob. for Laplace eqtn in another: (§5.3)



$$f(z) = \frac{z-i}{z+i}$$

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

adjoint $\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$



$$g(w) = \frac{iw+i}{-w+1} = \frac{i(w+1)}{-w+1} = -i\left(\frac{w+1}{w-1}\right)$$

<u>Example</u> (see HW II last week). Find harmonic fcn u in disk

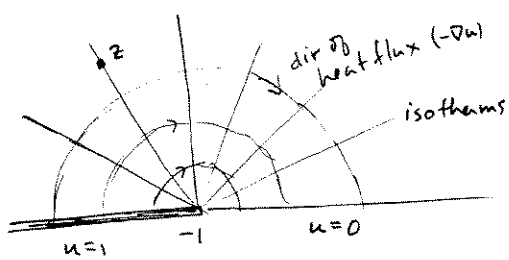
$u = 1$ for $e^{i\theta}$ $0<\theta<\pi/2$

$u = 0$ for $e^{i\theta}$ $\pi/2<\theta<2\pi$

$g(w)$ transforms the quarter arc with $0<\theta<\pi/2$ to $(-\infty,-1)$

Consider

$$u(z) = \frac{\arg(z--1)}{\pi} = \frac{\arg(z+1)}{\pi}$$

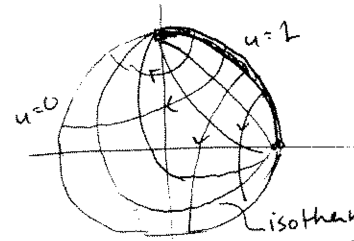


Then $u(g(w)) = u\left(-i\left(\frac{w+1}{w-1}\right)\right)$ solves problem in disk,

because

harmonic ∘ (analytic) = harmonic

(remember why?).



isotherms are circular arcs by LFT property!

So are heat flux curves!

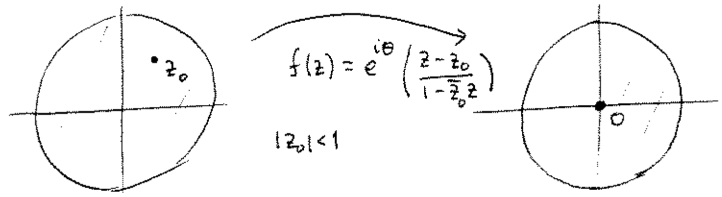
$$\text{note, } u\circ g(w) = \frac{1}{\pi}\arg\left(\frac{-i(w+1)}{w-1}+1\right)$$

$$= \frac{1}{\pi}\arg\left(\frac{-iw-i+w-1}{w-1}\right)$$

$$= \frac{1}{\pi}\arg\left(\frac{(w-i)(1-i)}{(w-1)}\right) = \frac{1}{\pi}\left(\arg\left(\frac{w-i}{w-1}\right) + \arg(1-i)\right)$$

$$= \frac{1}{\pi}\left(\arg\left(\frac{w-i}{w-1}\right) - \pi/4\right) \checkmark$$

- The Möbius transformations of the unit disk: (also page 4 Fri.)



Note, if $|z|=1$ then $|f(z)| = \frac{|z-z_0|}{|z|\,|\bar{z}-\bar{z}_0|} = 1$!

So f maps unit circle to unit circle & z_0 to 0

thus Disk to Disk! (see last page of notes for general discussion, or give special proof in this case)

<u>Theorem</u> : these are <u>all</u> conformal bijections of disk to itself

<u>pf</u> : Let $T: D \to D$ a conformal bijection.

$\exists\, z_0 \in D$ s.t. $T(z_0) = 0$.

Let $\arg(T'(z_0)) = \theta_0$

consider $f(z) = e^{i\theta_1}\left(\frac{z-z_0}{1-\bar{z}_0 z}\right)$ where θ_1 is chosen so that $\arg(f'(z_0)) = \theta_0$ too

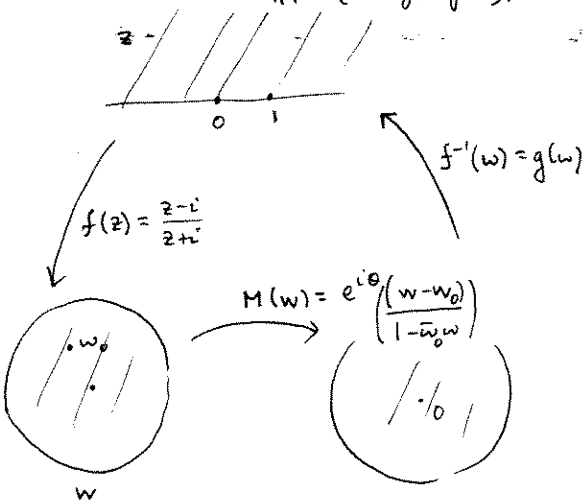
Then $f^{-1}\circ T : D \to D$, $z_0 \to z_0$

$\arg(f^{-1}\circ T)'(z_0) = 0 \Rightarrow f^{-1}\circ T = id$ by uniqueness in RMT

$\Rightarrow T = f$.

- $SL(2,\mathbb{R})$ acting on upper half plane.

Consider $\mathbb{H} = \{(x+iy) \mid y>0\}$.



every conformal bij $\mathbb{H}\to\mathbb{H}$ is a composition LFT

$f^{-1}\circ M \circ f$

(If $T: \mathbb{H}\to\mathbb{H}$, then

$f\circ T\circ f^{-1} = M$ by top of page)

alternately, one can see that these LFT's can be written as

$$T(z) = \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{R}$, $ad-bc = +1$ $\Big\}$ $\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in SL(2,\mathbb{R})$

special linear group of ~~with real coeffs.~~ real 2×2 matrices having $\det = +1$.

$\Rightarrow \mathbb{R}\to\mathbb{R}$

$\Rightarrow i \longmapsto \mathbb{H}$

$$T(i) = \frac{ai+b}{ci+d}\cdot\frac{-ci+d}{-ci+d} = \frac{ac+bd+(ad-bc)i}{c^2+d^2}$$

There's a nice (and fundamental) story in geometry, having to do with these transformations ...

In <u>Riemannian Geometry</u> one studies manifolds M which have (infinitesimal) elements of arc-length, so that one can study lengths of curves. (which leads to a lot more?)

An <u>isometry</u> ~~preserves~~ $T: M \to M$ preserves all curve lengths.

<u>example</u> $\mathbb{R}^2$, $ds = \sqrt{dx^2 + dy^2}$ (so $\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}\, dt$).

<u>Theorem</u>: The only orientation preserving isometries of $\mathbb{R}^2$ are

$$T(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x} + \vec{b} = T_{\theta,b}(\vec{x})$$

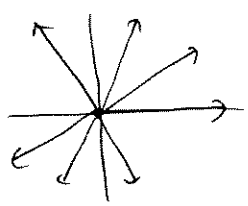
"proof": let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be any isometry. (which preserves orientation)

Since T preserves all curve lengths, by looking at short curve we deduce it preserves arclength elements, so the derivative matrix of T at any point must be a rotation matrix (but, a prior, this rotation could change as you change pts)

Compose T with some $T_{\theta,b}$ so that

$$\left.\begin{aligned} (T \circ T_{\theta,b})(0) &= 0 \\ (\qquad\quad)'(0) &= I \end{aligned}\right\}$$ $T \circ T_{\theta,b}$ is isom which fixes origin & tangent vector at origin.

shortest lines (paths) btwn pts $\to$ shortest lines (paths) btwn pts.



$\Rightarrow$ pos. x-axis $\to$ a curve which minimizes dist btwn pts on it, with
$0 \to 0$
x-dir $\to$ x-dir

$\Rightarrow$ pos x-axis $\to$ pos x-axis, as identity

$\Rightarrow$ each ray from origin $\to$ itself, as identity!

$\Rightarrow T \circ T_{\theta,b} = id$

$\Rightarrow T = T_{\theta,b}^{-1}$ ∎

<u>example</u> $S^2 \subset \mathbb{R}^3$

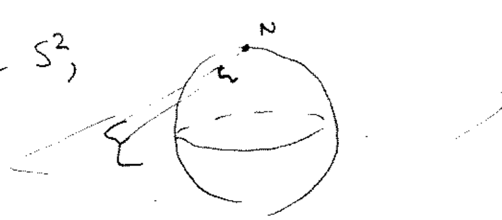
$$S^2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } x^2 + y^2 + z^2 = 1 \right\}$$

by a completely analogous argument, one can show that the only orient. pres. isoms of S^2 are the 3×3 rotation matrices (called $S.O.(3)$)

(i.e. compose with such a rot. matrix, to fix north pole & directions at north pole.).

By the way, using a stereographic chart for S^2,

$$\boxed{ds = \left(\frac{2}{1 + x^2 + y^2}\right)\sqrt{dx^2 + dy^2}}$$



There is one more model space !!

$\mathbb{H}^2$ = upper half space (Hyperbolic space)

$$ds = \frac{1}{y}\sqrt{dx^2+dy^2}$$

and the isometries are

$$T(z) = \frac{az+b}{cz+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$$

generated by $T_1(z) = \lambda z$, $\lambda > 0$ $\left(\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \right)$

$T_2(z) = z + b$ $b \in \mathbb{R}$ $\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right)$

$T_3(z) = -\frac{1}{z}$ $\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$

"straight forward" to show T_1, T_2, T_3 are isoms.

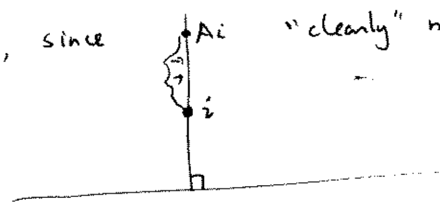
every orientation preserving isom must be conformal $\Rightarrow$ These are all.

e.g. $(T_1 \circ \gamma)'(t) = \lambda \gamma'(t)$

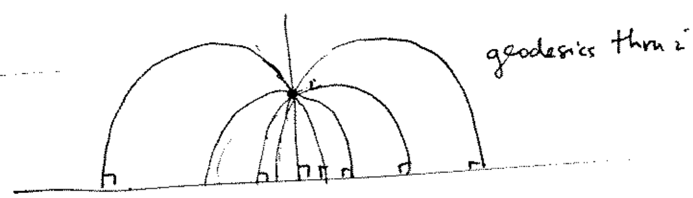
if $\gamma(t) = x_0 + iy_0$

$$\|\gamma'(t)\|_{\mathbb{H}} = \frac{\|\gamma'(t)\|_{\mathbb{E}}}{y_0} \qquad \|(T_1\circ\gamma)'(t)\|_{\mathbb{H}} = \frac{\|\lambda\gamma'(t)\|_{\mathbb{E}}}{\lambda y_0} = \|\gamma'(t)\|_{\mathbb{H}}.$$

Then, since

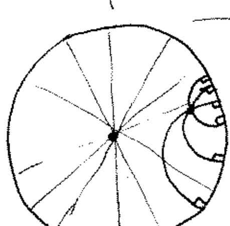


"clearly" minimizes length btwn i & Ai, all its images under isoms are also "geodesics"



another model (chart) for this space is the Poincaré disk $\mathbb{D} = \{z \mid |z|<1\}$

$$\boxed{ds = \frac{2}{1-x^2-y^2}\sqrt{dx^2+dy^2}}$$ (Compare to ds for sphere chart!)



<u>and</u> the page 1 $f(z), g(w)$ are isometries between $\mathbb{H}^2$ & $\mathbb{D}^2$.

(the isometries of $\mathbb{D}$ are the Möbius transformations)

A nifty theorem to justify our carelessness in conformal map checking:

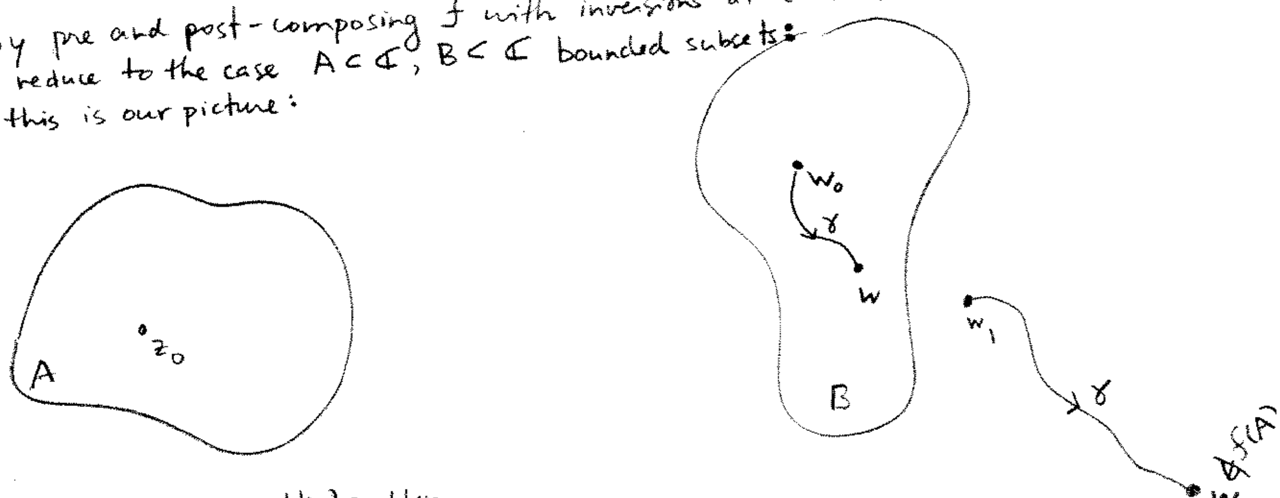
<u>**Thm**</u> Let A, $S^2_{\mathbb{R}} \setminus \overline{A}$ be connected (A open)
B, $S^2_{\mathbb{R}} \setminus \overline{B}$ connected (")

$f : A \to S^2_{\mathbb{R}}$ conformal
$f : \overline{A} \to S^2_{\mathbb{R}}$ continuous
$f(\partial A) \subset \partial B$
$\exists\, z_0 \in A$ s.t. $f(z_0) \in B$

Then $\begin{cases} f(A) = B \\ f(\partial A) = \partial B \end{cases}$, so if f is 1-1 then A and B are conformally equivalent.

<u>Remark</u>: So, roughly speaking, <u>conformal</u> maps can't leak over their edges. Compare this to our discussions during homotopy days, when keeping the homotopy in the domain <u>had</u> to be checked, even the the bdry map was in domain.

<u>Proof</u>: By pre and post-composing f with inversions at $z^* \notin \overline{A}$, $w^* \in \overline{B}$ we may reduce to the case $A \subset \mathbb{C}$, $B \subset \mathbb{C}$ bounded subsets.
So this is our picture:



(1) $f(A)$ is open: this was a HW problem in chapter 1; idea: Let $z \in A$. $f'(z) \neq 0 \Rightarrow$ local inverse fcn $\Rightarrow \exists\, D(f(z), \epsilon) \subset f(A)$. ■
$[f(z) \in f(A)]$

(2) $B \subset f(A)$: pick $w \in B$; $\gamma : [a,b] \to B$
$\gamma(a) = w_0$
$\gamma(b) = w$
Then $f(A) \cap \gamma = f(\overline{A}) \cap \gamma$, (since $f(\partial A) \cap B = \emptyset$)
so this set is open and closed in γ,
i.e. $\{t \in [a,b] \mid \gamma(t) \in f(A)\}$ is open and closed.
since $a \in$ this set, set is entire $[a,b]$
$w = \gamma(b) \in f(A)$ ■

(3) $f(A) \subset B$; Suffices to show $f(A) \subset \overline{B}$ since $f(A)$ is open.
Let $w_1 \in \mathbb{C} \setminus B$. $|w| > \max_{z \in \overline{A}} |f(z)|$, so $w \notin f(A)$
γ a path in $\mathbb{C} \setminus B$ connecting w_1 to w. Then $f(A) \cap \gamma = f(\overline{A}) \cap \gamma$ is open and closed subset of γ.
Since $w \notin f(A)$, set $f(A) \cap \gamma$ is empty $\Rightarrow w_1 \notin f(A)$

(4) hence $f(A) = B$.
$\overline{f(A)} = \overline{B}$, i.e. $f(A) \cup f(\partial A) = B \cup \partial B \Rightarrow f(\partial A) = \partial B$ ■