Math 4200
Fri Dec 2

Last HW assignment! Due 12/9 (or earlier)

5.1 10, 11, 12
5.2 4a, 6, 7, 10, 24, 26, 33, 34

On Wednesday we proved the uniqueness part of the Riemann Mapping Theorem (see more advanced text for $\exists$)

<u>Theorem</u>: Let $A \neq \mathbb{C}$ open, simply connected $z_0 \in A$
$B \neq \mathbb{C}$ " " $w_0 \in B$

then $\exists!$ conformal bijection $f: A \to B$
s.t. $f(z_0) = w_0$
$f'(z_0) > 0$ $\leftarrow$ (we could've specified $\arg f'(z_0)$ to be anything)

We looked at 3 examples for which we identified some conformal bijections, but in the light of RMT we were missing some (3 real degrees of freedom).

Using <u>fractional</u> <u>linear</u> <u>transformations</u> (FLT's)
we will be able to find all conformal transformations of the disk $\{z \mid |z|<1\} = D$ to itself, and this would let us find all bijections in the 3 earlier examples ~~(and, actua~~

an FLT is any transformation

$$f(z) = \frac{az+b}{cz+d} \qquad \text{with } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \quad (\text{i.e. } f \text{ is not constant})$$

$$a, b, c, d \in \mathbb{C}$$

Properties of FLT:

1. If $f(z) = \frac{az+b}{cz+d}$, $g(w) = \frac{\alpha w+\beta}{\gamma w+\delta}$

then $g \circ f(z) = \dfrac{\alpha\left[\frac{az+b}{cz+d}\right] + \beta}{\gamma\left[\frac{az+b}{cz+d}\right] + \delta} \cdot \dfrac{(cz+d)}{(cz+d)}$

$$= \frac{z(\alpha a + \beta c) + (\alpha b + \beta d)}{z(\gamma a + \delta c) + (\gamma b + \delta d)}$$

Define "the" matrix of f to be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (defined up to non-zero scalar multiple, which does not change value of f)

<u>Deduce</u>: f, g FLT's $\Rightarrow g \circ f$ FLT <u>and</u> has matrix $= [g\circ f] = [g][f]$.

In the language of group theory, we have a homomorphism from in group of invertible 2×2 matrices to the group of conformal bijections from $S^2_R \to S^2_R$ (see 2)

$\hookrightarrow f'(z) = \dfrac{a(cz+d) - c(az+b)}{(cz+d)^2}$

$= \dfrac{ad-bc}{(cz+d)^2}$; check $z = \infty, -\frac{d}{c}$ separately.

$\neq 0$.

2. Hence if $f(z) = \frac{az+b}{cz+d}$

then $f^{-1}(w) = \dfrac{dw-b}{-cw+a}$! (Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$).

$\Rightarrow$ conformal:

$f^{-1}(f(z)) = z \Rightarrow (f^{-1})'(w)\, f'(z) = 1 \Rightarrow f'(z) \neq 0$ (check $z = \infty, -\frac{d}{c}$ separately) using 2 3, & 4

$\Rightarrow$ bijection on S^2_R:

if $c = 0$ clear

if $c \neq 0$, $\exists f^{-1} \Rightarrow$ $\left.\begin{array}{l} f: \mathbb{C}\setminus\{-\frac{d}{c}\} \to \mathbb{C}\setminus\{\frac{a}{c}\} \text{ bijection} \\ f(-\frac{d}{c}) = \infty \\ f(\infty) = (\frac{a}{c}) \end{array}\right\}$ $f: S^2_R \to S^2_R$ bij.

3. FLT's map circles and lines to circles or lines.

<u>pf</u>: every FLT is a composition of more elementary FLT, of form

(1) $f_1(z) = z + a$ (translation)

(2) $f_2(z) = cz$ (dilation rotation)

(3) $f_3(z) = \frac{1}{z}$ (inversion). (HW 6 5.2 #11)

types (1), (2) clearly map lines $\to$ lines, circles $\to$ circles.

So need only check (3). But if

$$Ax + By + C(x^2+y^2) = D$$

and $z = x+iy$ then $\frac{1}{z} = \frac{x-iy}{x^2+y^2} = u + iv$

so

<u>example</u>: $f(z) = \dfrac{z+i}{2z+3}$:

$\frac{1}{2} = \dfrac{z+3/2}{2(z+3/2)}$

so $f(z) = \frac{1}{2} + \dfrac{i-3/2}{2z+3}$

$z \to 2z \to 2z+3 \to \frac{1}{2z+3} \to (i-\frac{3}{2})\frac{1}{2z+3} \to f$

$$A\frac{x}{x^2+y^2} + B\frac{y}{x^2+y^2} + C\left(\frac{x^2+y^2}{x^2+y^2}\right) = \frac{D}{x^2+y^2}$$

$$Au - Bv + C = D(u^2+v^2)$$ circle or line!

A lot of geometry follows from (3); since 3 distinct points uniquely determine a line or circle.

example $f(z) = \dfrac{z-i}{z+i}$

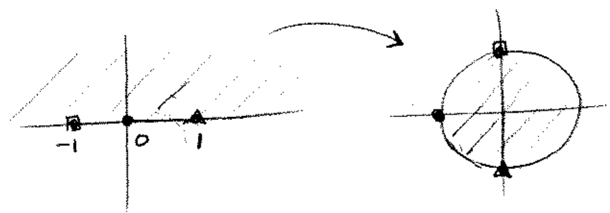
$1 \to \frac{1-i}{1+i} = -i$

$0 \to -1$

$-1 \to \frac{-1-i}{-1+i} = i$

↑ on real axis ↑ on unit circle

so



gives a conformal map from upper half plane to unit disk.

if $y>0$

$\left|\frac{x+i(y-1)}{x+i(y+1)}\right| < 1$

(4) 3-point property:

$\exists!$ LFT f s.t. $f(z_j) = w_j$, $j=1,2,3$, $\{z_1,z_2,z_3\}$, $\{w_1,w_2,w_3\}$ distinct

<u>proof</u> if want $T(z_1)=0$

$T(z_2)=\infty$

$T(z_3)=1$

use $T(z) = \left(\frac{z-z_1}{z-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right)$, and T is unique!

similarly for $S(w_1)=0$

$S(w_2)=\infty$

$S(w_3)=1$

$$S(w) = \left(\frac{w-w_1}{w-w_2}\right)\left(\frac{w_3-w_2}{w_3-w_1}\right)$$

so $f = S^{-1}\circ T$ works.

uniqueness follows from diagram (uniqueness of S & T)

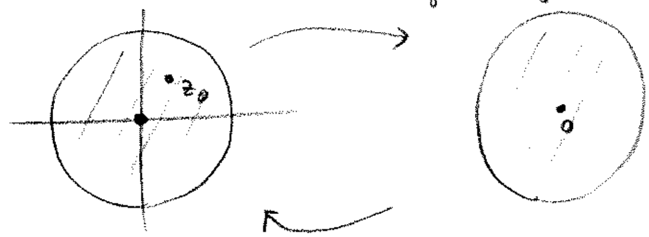
$$\begin{tikzcd} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} \arrow[rr, "f"] \arrow[dr, "T"'] & & \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \arrow[dl, "S"] \\ & \begin{Bmatrix} 0 \\ \infty \\ 1 \end{Bmatrix} & \end{tikzcd}$$

by uniqueness $T = S\circ f$

(so $f = S^{-1}\circ T$)

examples

$$f(z) = \frac{z - z_0}{1 - \bar{z}_0 z} = \frac{z - z_0}{-\bar{z}_0 z + 1}$$



$$f^{-1}(w) = \frac{w + z_0}{\bar{z}_0 w + 1}$$

note $\frac{\bar{z} - \bar{z}_0}{\bar{z}(\bar{z} - \bar{z}_0)} = \overline{f(z)}$ on $|z| = 1$

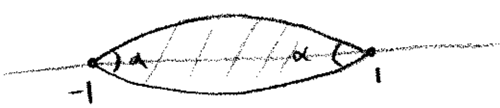
so $|f(z)| = 1$ there

Notice, every conformal bijection $g : D \to D$ maps some z_0 to 0, so $g(z)$ can be compared to

$$f(z) = e^{i\theta}\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right) \leftarrow \text{"Möbius transformation"}$$

where θ is chosen so that $\arg f'(z_0) = \arg g'(z_0)$.

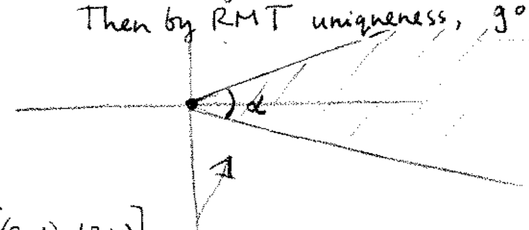
Then by RMT uniqueness, $g \circ f^{-1} = id$ so $f = g$



(circular arcs)

$$T_1(z) = \frac{z+1}{z-1}(-1)$$

$-1 \to 0$
$0 \to 1$
$1 \to \infty$



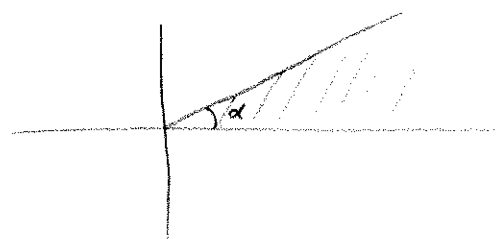
$$T_1'(z) = \frac{(-1)\left[(z-1) - (z+1)\right]}{(z-1)^2} = \frac{2}{(z-1)^2}$$

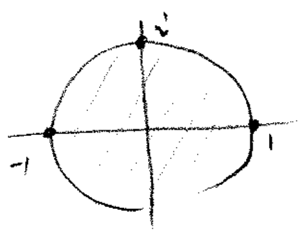
$$T_1'(-1) = \frac{1}{2}$$

(shows how tang vectors rotate)

?

$$T_2(w) = e^{i\alpha/2} w$$



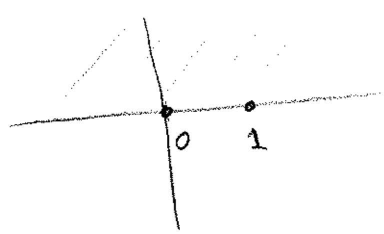


(page 4 example)

$$T_4(w) = \frac{w - i}{w + i}$$

$0 \to -1$
$1 \to -i$
$\infty \to 1$

$$T_3(z) = z^{\frac{\pi}{\alpha}} = e^{\frac{\pi}{\alpha}\log z}$$



$f = T_4 \circ T_3 \circ T_2 \circ T_1$ will do.

(Have freedom from top example too)