Math 4200
Wed 31 Aug

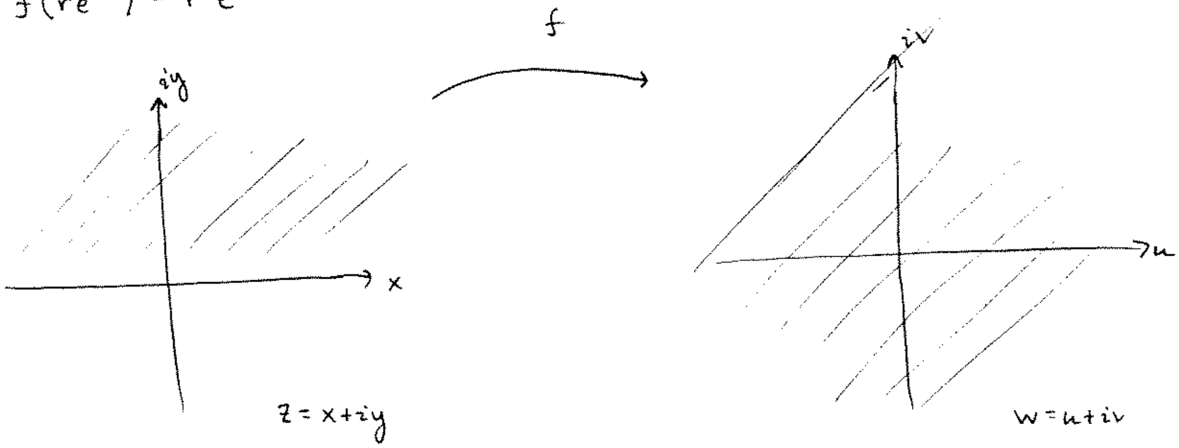
remember HW due Fri
at start of class!
optional problem session Thurs (tomorrow)
JWB 335 10:45 - 12:05

§1.3 : "Elementary" complex differentiable functions, and possible choices for their "inverse" fcns.

<u>Example</u>

$f(z) = z^2$

$f(re^{i\theta}) = r^2 e^{2i\theta}$



$z^2 = w$

$z = \pm\sqrt{w}$

↑

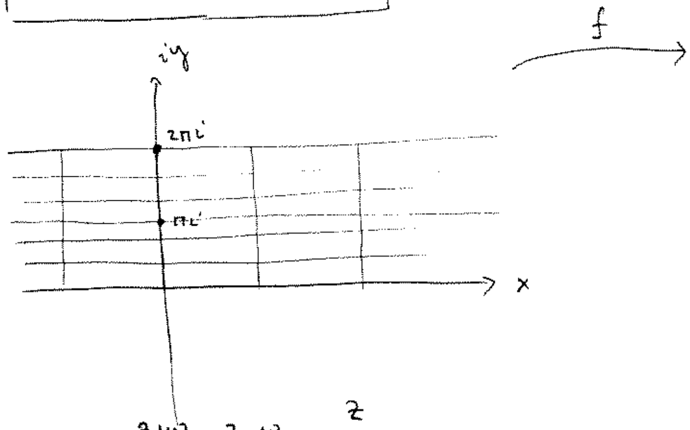
2 different choices possible.

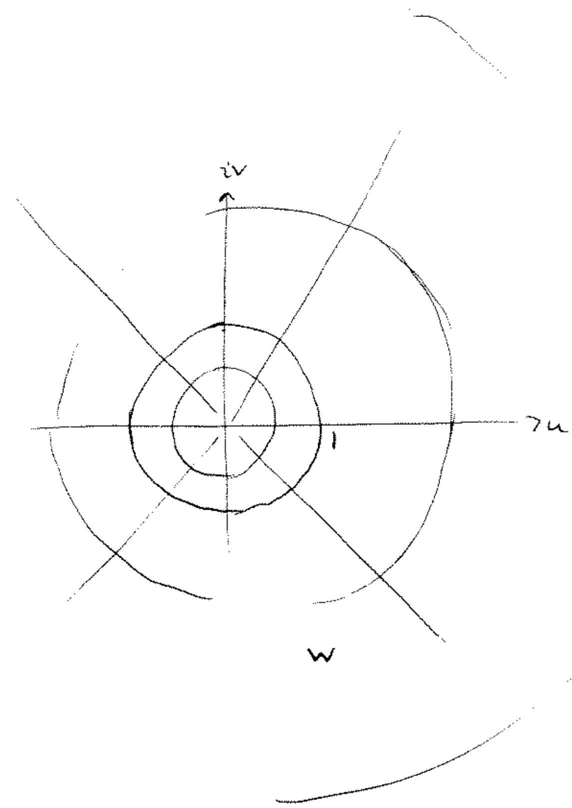
$f(z) = z^3$

$f(re^{i\theta}) = r^3 e^{i(3\theta)}$

Example $\quad z = x + iy$

$$\boxed{f(z) = e^{z} := e^{x} e^{iy}}$$





property: $e^{z+w} = e^{z} e^{w}$
did we check this?

inverse fcn?

$$e^{z} = w$$

$$e^{x}(\cos y + i \sin y) = u + iv$$

$$\Longleftrightarrow \begin{cases} |w| = e^{x} \Rightarrow x = \log|w| \\ y = \arg(w) \end{cases}$$

so (switching letters),

$$\boxed{\log z := \log|z| + i \arg z}$$

if, for example, we restrict $0 \le \arg z < 2\pi$
then we get a "branch" of the log function which
maps $\mathbb{C} \setminus \{0\}$ 1-1 and onto the (half open) strip above

and $\log(e^{z}) = z$, $\quad e^{\log z} = z$ with these restrictions

Many other "branch" choices exist!

verify!

also, log property:

$$\log zw = \log z + \log w$$
(up to a multiple of 2π in imag. part)

Trig fcns

If x is real, $e^{ix} = \cos x + i\sin x$ Eqtn 1

$e^{-ix} = \cos x - i\sin x$ Eqtn 2

Also

$\frac{\text{Eqtn1} + \text{Eqtn2}}{2} \Rightarrow \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$; $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\frac{\text{Eqtn1} - \text{Eqtn2}}{2i} \Rightarrow \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$; $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

$\cos z := \frac{1}{2}(e^{iz} + e^{-iz})$ $\cosh z := \frac{1}{2}(e^{z} + e^{-z}) = \cos(iz)$

$\sin z := \frac{1}{2i}(e^{iz} - e^{-iz})$ $\sinh z := \frac{1}{2}(e^{z} - e^{-z}) = \frac{1}{i}\sin(iz)$

or

$$\left(\begin{array}{l} \cos z = \cosh(iz) \\ \sin z = \frac{1}{i}\sinh(iz) \end{array} \right).$$

Trig identities hold! (So also trigh identities)

$\sin^2 z + \cos^2 z = 1$

$\sin(z+w) = \sin z \cos w + \cos z \sin w$

$\cos(z+w) = \cos z \cos w - \sin z \sin w$

trigh...

$\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1 \ldots$

power functions

For a given branch of log, define

$$z^w := e^{w \log z} \qquad \text{(uniquely defines } z^w \text{ for the given branch of log)}$$

In general z^w is multiple-valued

- If n is integer, $z^n = e^{n \log z} = e^{n(\log|z| + i \overset{\pm 2\pi k}{\arg z})}$

$$= \left[e^{\log|z| + i(\arg z + 2\pi k)}\right]^n \qquad (e^{z+w} = e^z e^w)$$

$$= z^n \qquad \text{since } e^{2\pi i k} = \cos 2\pi k + i \sin 2\pi k$$
$$= 1 \text{ for } k \text{ an integer}$$

- If $w = \frac{p}{q}$ is a rational in lowest terms, (p, q no common divisors)

then $z^{p/q}$ is one of the possible q values of (previous) p/q^{th} roots, i.e.

$$z = re^{i\theta}$$
$$z^{p/q} = r^{p/q} e^{ip(\cdot \frac{1}{q}(\theta + 2\pi k))} \qquad k = 0, 1, \dots q-1$$

you can check this on your own!

- If w is irrational,

z^w has ∞'ly many possible values, depending on log branch

Inverse trig fcns!

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ is a composition of

$w = f(z) = e^{iz}$

with $\zeta = g(\omega) = \frac{1}{2}(\omega + \frac{1}{\omega})$

$$\begin{tikzcd} z \arrow[r, bend left, "f"] & e^{iz} = w \arrow[l, bend left, dashed, "f^{-1}"] \arrow[r, bend left, "g"] & \zeta = \frac{1}{2}(\omega + \frac{1}{\omega}) \arrow[l, bend left, dashed, "g^{-1}"] \end{tikzcd}$$

$\zeta = \frac{1}{2}(\omega + \frac{1}{\omega})$

$2\zeta = \frac{\omega^2+1}{\omega}$

$\omega^2 - 2\zeta\omega + 1 = 0$

$\omega = \cancel{2}\zeta \pm \sqrt{\zeta^2 - 1}$ (2 choice

$g^{-1}(\zeta) = \zeta \pm \sqrt{\zeta^2-1}$

$e^{iz} = \omega$

$iz = \log w$ (branch choice!)

$z = -i \log w$

$f^{-1}(w) = -i \log w$

So!!

$\cos^{-1}(\zeta) = f^{-1} g^{-1}(\zeta)$

$= -i \log(\zeta \pm \sqrt{\zeta^2-1}) \ldots$