This course is about Calculus for complex differentiable (also called analytic) functions... the def. of diffible looks just like it did in Calc, but has wildly stronger consequences because our variables are complex.

\[ f: \mathbb{C} \to \mathbb{C}, \quad A \subseteq \mathbb{C} \text{ open} \]
\[ z_0 \in A \]

\[ f \text{ is (complex) differentiable at } z_0 \text{ iff } \]
\[ \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \text{ exists} \]

\[ \iff f(z_0 + h) = f(z_0) + f'(z_0) h + o(h), \]
\[ \lim_{h \to 0} \frac{o(h)}{h} = 0 \]

Write \( z = x + iy \)
\[ f(z) = u(z) + iv(z) \]

Then \( f(z) \) corresponds to an equivalent real-valued function
\[ F(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}, \quad F : A \to \mathbb{R}^2 \]

Math 3220 def of \( F \) diffible at \([x_0, y_0]\)

\[ F(x_0 + h_1, y_0 + h_2) = F(x_0, y_0) + \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + o(h) \]
\[ \lim_{h \to 0} \|o(h)\| = 0 \]

How are these two notions related?
We shall see that complex diffible is a special case of real-diffible.
Example

\[ f(z) = z^2 \]

\[ \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2zh + h^2}{h} = 2z \]

\[ f(re^{\iota \theta}) = r^2 e^{2\iota \theta} \]

Approximation formula:

Let \( z_0 = 1 + i \)

\[ f'(z_0) = 2z_0 = 2 + 2i \]

\[ f(z_0 + h) = f(z_0) + f'(z_0) h + e(h) \]

affine approx to \( f \)

Note: \( 2 + 2i = 2\sqrt{2} e^{\iota \pi/4} \)

Use L-box to represent affine approximation

- \( z_0 + h \) is mapped to \( f(z_0) + h \), where \( h \) is obtained from \( h \) by rotating it \( \pi/2 \) and dilating it by a factor of \( 2\sqrt{2} \)

Real form:

\[ F(y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \]

\[ DF(y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \]

\[ F(1+h_1 + h_2) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(k) \]

\[ = 2\sqrt{2} \begin{bmatrix} \frac{h_2}{\sqrt{2}} \\ \frac{h_1}{\sqrt{2}} \end{bmatrix} \]

rotation
Example

\[ f(z) = e^z \]

\[ e^{x+iy} := e^x e^{iy} = e^x (\cos y + i \sin y) \]

Note,
\[ e^{zi} = e^x e^{yi} \]

check: \( (x_1 + iy_1) + (x_2 + iy_2) := e^{x_1 + iy_1} e^{i(y_1 + y_2)} \)
\[ = e^{x_1} e^{x_2} e^{iy_1} e^{iy_2} \]
\[ = e^{x_1 + iy_1} e^{x_2 + iy_2} \]

(checked this before)

\[ \lim_{h \to 0} \frac{e^{zh} - e^z}{h} = \lim_{h \to 0} e^z \left( \frac{e^{h} - 1}{h} \right) = e^z \]

(?!)

(you could do this, we'll do it another way later)

\[ f'(z) = e^z \]

\[ f(z) = e^z e^{iy} \]
\[ z_0 = \frac{\pi}{2} + i \frac{\pi}{2} \]

\[ f(z_0) = i e^{\frac{\pi}{2}} \]

\[ f'(z_0) = \frac{d}{dz} f(z) \]

\[ f(z_0 + h) = f(z_0) + i e^{\frac{\pi}{2} h} e^{i h} \]

related fns: \( \log z \), \( \cos z \), \( \sin z \), \( z^2 \)

**Example:**

\[ f(z) = \frac{z - i}{z + i} \]

\[ f'(z) = \frac{(z + i) - (z - i)}{(z + i)^2} = \frac{2i}{(z + i)^2} \]

The upper half plane is mapped to the unit disk.
**Theorem** \( \text{let } f(x+iy) = u(x,y) + iv(x,y) \) correspond as on page 1, on the open set \( A \subset \mathbb{C} \) \( \text{let } z_0 \in A \).

Then \( f \) is complex differentiable at \( z_0 = x_0 + iy_0 \)

iff

\( F \) is real differentiable at \( \left[ \frac{x_0}{y_0} \right] \), with \( \text{DF} \left( \frac{x_0}{y_0} \right) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \)

a rotation-dilation matrix, i.e. \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \)

The equations \( \begin{bmatrix} u_x = v_y \\ u_y = -v_x \end{bmatrix} \) are called the Cauchy-Riemann eqtns.

\[ \begin{align*}
\text{Proof:} & \quad \Rightarrow \quad f \text{ complex differentiable at } z_0 \text{ implies} \\
& \quad f(z_0 + eh) = f(z_0) + f'(z_0)h + e(h) \quad \text{, } \frac{e(h)}{h} \to 0 \text{ as } h \to 0.
\end{align*} \]

write \( f'(z_0) = a + bi \)

then \( f'(z_0)h = (a + bi)(h_1 + ih_2) \)

\[ = ah_1 - bh_2 + i(gh_1 - ah_2) \]

so \( F \left( \frac{x_0 + k_1}{y_0 + k_2} \right) = F \left( \frac{x_0}{y_0} \right) + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(h) \)

where \( \| E(h) \| = \left\| \frac{e(h)}{h} \right\| \to 0 \) as \( h \to 0. \)

\[ \Leftarrow: \text{If} \]

\[ F \left( \frac{x_0 + k_1}{y_0 + k_2} \right) = F \left( \frac{x_0}{y_0} \right) + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(h) \quad \text{with} \quad \| E(h) \| \to 0 \quad \text{as} \quad h \to 0 \]

then

\[ u(z_0 + k_1) + iv(z_0 + k_2) = u(z_0) + iv(z_0) + (a + bi)(h_1 + ih_2) + e(h), \quad \| \frac{e(h)}{h} \| \to 0 \quad \text{as} \quad h \to 0 \]

\( f(z_0 + k_1) = f(z_0) + (a + bi)k_1 + e(k_1) \)

\( f(z_0 + k_1) - f(z_0) = a + bi + \frac{e(k_1)}{k_1} \) \[ \blacksquare \]