Math 4200
Monday 29 Aug

This course is about Calculus for complex differentiable (also called analytic) functions ... the def. of diff'ble looks just like it did in Calc, but has wildly stronger consequences because our variables are complex.

$f: A \to \mathbb{C}$, $A \subset \mathbb{C}$ open
$z_0 \in A$

f is <u>(complex) differentiable</u> at z_0 iff

$$\boxed{\begin{array}{l} \lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} := f'(z_0) \quad \text{exists} \\[2em] \text{iff} \quad f(z_0+h) = f(z_0) + f'(z_0)h + e(h), \\ \qquad\qquad\qquad \lim_{h\to 0} \dfrac{e(h)}{h} = 0 \end{array}}$$

Write $z = x+iy$
$f(z) = u(z) + i v(z)$

Then $f(z)$ corresponds to an equivalent real-valued fcn

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \quad F: A \to \mathbb{R}^2$$

Math 3220 def of F <u>diff'ble</u> at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

deriv. matrix evaluated at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

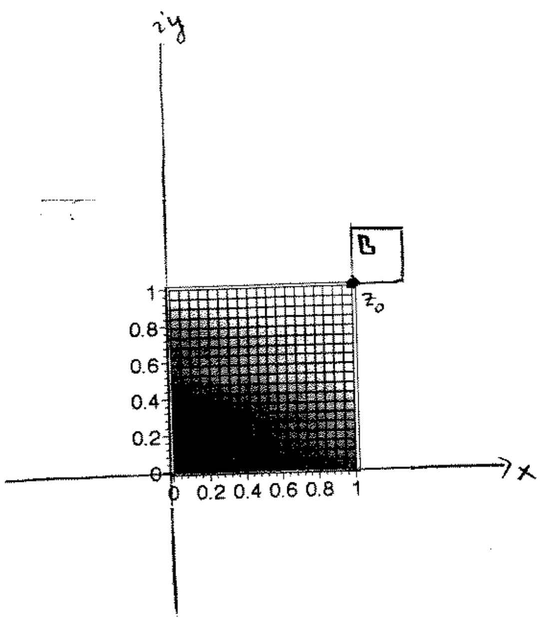
$$\boxed{\begin{array}{l} F\begin{pmatrix} x_0+h_1 \\ y_0+h_2 \end{pmatrix} = F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + E(h), \\ \qquad\qquad\qquad \lim_{h\to 0} \dfrac{\|E(h)\|}{\|h\|} = 0 \end{array}}$$

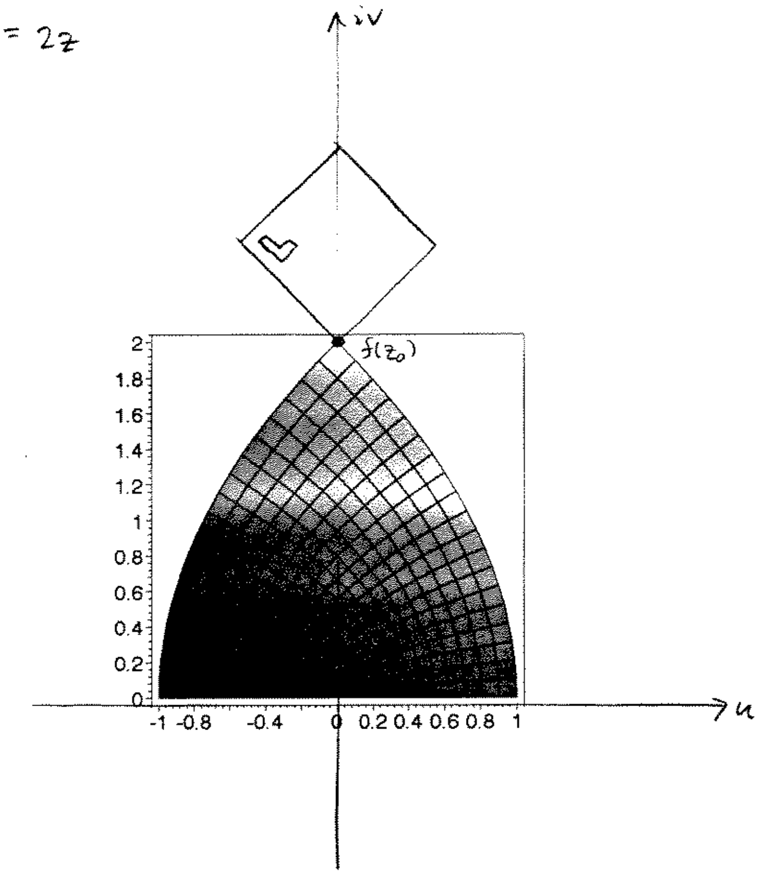
How are these two notions related?
We shall see that complex diff'ble is a special case of real-diff'ble.

Example

$f(z) = z^2$

$$\lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2hz + h^2}{h} = 2z$$





$f(z) = z^2$

$f(re^{i\theta}) = r^2 e^{2i\theta}$

Approximation formula:

let $z_0 = 1+i$

$f'(z_0) = 2z_0 = 2+2i$

$$f(z_0+h) = f(z_0) + f'(z_0)h + e(h)$$
$$= \underbrace{2i + (2+2i)h}_{\text{affine approx to } f} + e(h)$$

note: $2+2i = 2\sqrt{2}\, e^{i\pi/4}$

Use L-box to represent affine approximation

- z_0+h is mapped to $f(z_0)+k$, where k is obtained from h by rotating it $\pi/4$ & dilating it a factor of $2\sqrt{2}$

Real form:

$$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2-y^2\\2xy\end{pmatrix}$$

$$DF\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}2x & -2y\\2y & 2x\end{bmatrix}$$

$$F\begin{pmatrix}1+h_1\\1+h_2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} + \begin{bmatrix}2 & -2\\2 & 2\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix} + E(\vec{h})$$

$$= \underbrace{2\sqrt{2}}_{\text{dilation}} \underbrace{\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}}_{\text{rotation}}$$

Example

$$f(z) = e^z$$

$$e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)$$

Note,

$$e^{z+w} = e^z e^w$$

check: $e^{(x_1+iy_1)+(x_2+iy_2)} := e^{x_1+x_2} e^{i(y_1+y_2)}$

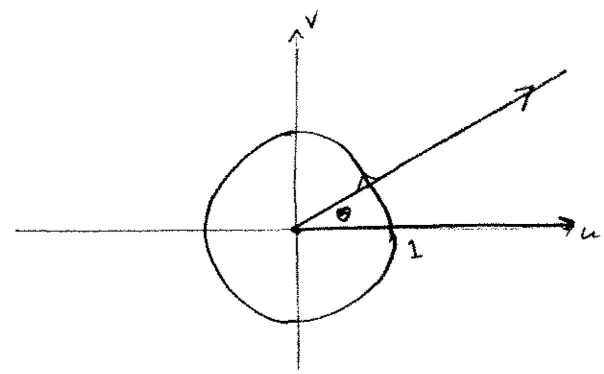
$= e^{x_1} e^{x_2} e^{iy_1} e^{iy_2}$ (checked this before)

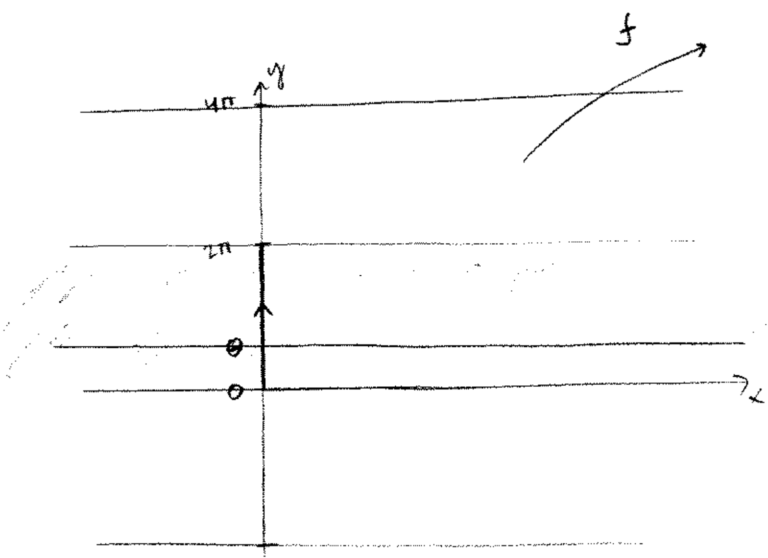
$= e^{x_1+iy_1} e^{x_2+iy_2}$

$$\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \to 0} e^z \left(\frac{e^h - 1}{h} \right) = e^z \qquad (?!)$$

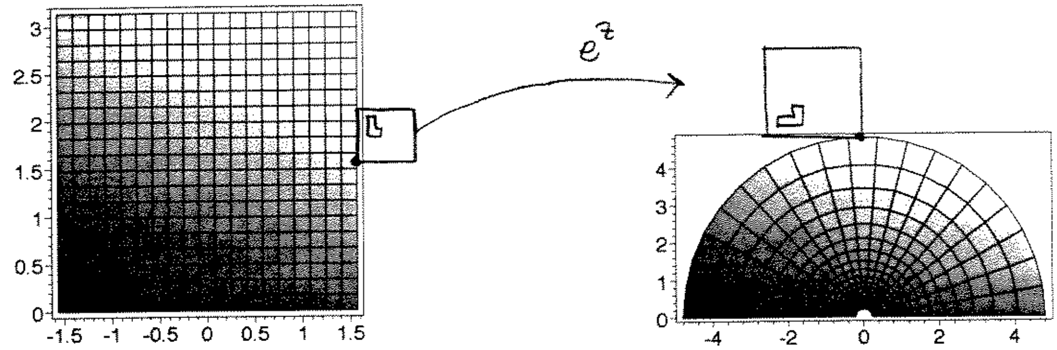
(you could do this, we'll do it another way later)

$$f'(z) = e^z$$





$$f(x+iy) = e^x e^{iy}$$



$z_0 = \frac{\pi}{2} + i\frac{\pi}{2}$

$f(z_0) = i e^{\pi/2}$

$f'(z_0) = $ ''

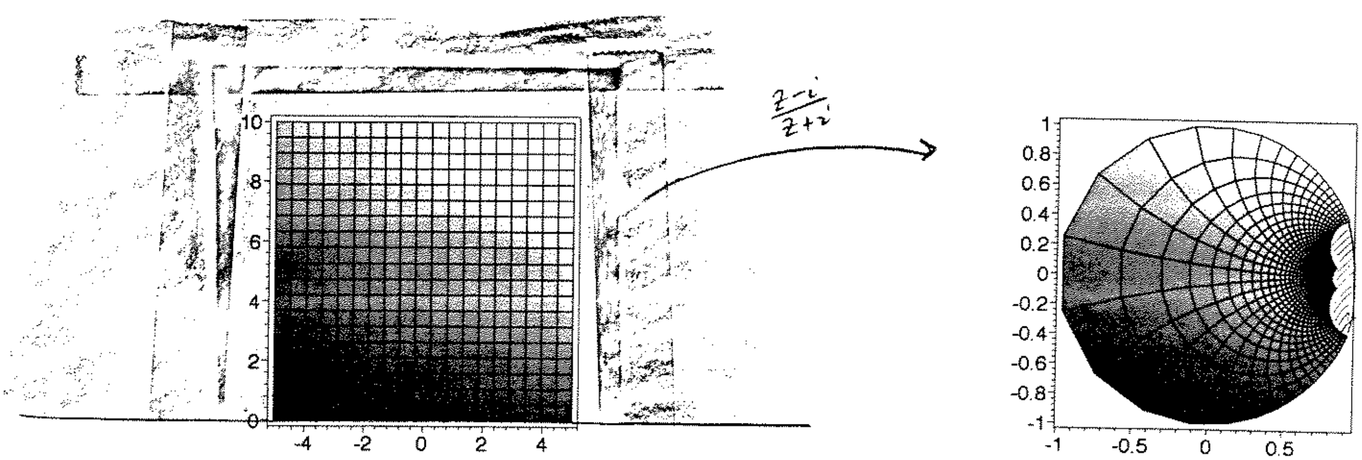
$f(z_0+h) = f(z_0) + i e^{\pi/2} h + e(h)$

rotate by $\pi/2$ | scale by $\pi/2$

related fcns: $\log z$, $\cos z$, $\sin z$, z^a

example: $f(z) = \frac{z-i}{z+i}$

$$f'(z) = \frac{(z+i)-(z-i)}{(z+i)^2} = \frac{2i}{(z+i)^2}$$



upper half plane is mapped to the unit disk.

<u>Theorem</u> Let $f(x+iy) = u+iv$ and $F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}$ correspond as on page 1, on the open set $A \subset \mathbb{C}$

Let $z_0 \in A$.

Then f is complex differentiable at $z_0 = x_0 + iy_0$

iff

F is real diffble at $\begin{bmatrix}x_0\\y_0\end{bmatrix}$, with $DF\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}$

a rotation-dilation matrix, i.e. $= \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

The equations $\boxed{\begin{matrix}u_x = v_y\\ u_y = -v_x\end{matrix}}$ are called the Cauchy-Riemann eqtns

<u>pf</u>: $\Rightarrow$ f complex diffble at z_0 implies

$$f(z_0+h) = f(z_0) + f'(z_0)h + e(h) \quad , \frac{e(h)}{h} \to 0 \text{ as } h \to 0.$$

write $f'(z_0) = a+bi$

then $f'(z_0)h = (a+bi)(h_1+ih_2)$

$= ah_1 - bh_2$
$\quad + i(bh_1 - ah_2)$

so $F\begin{pmatrix}x_0+h_1\\y_0+h_2\end{pmatrix} = F\begin{pmatrix}x_0\\y_0\end{pmatrix} + \begin{bmatrix}a & -b\\ b & -a\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix} + E(h)$

where $\frac{\|E(h)\|}{\|h\|} = \left|\frac{e(h)}{h}\right| \to 0$ as $h \to 0$. ∎

$\Leftarrow$: If

$F\begin{pmatrix}x_0+h_1\\y_0+h_2\end{pmatrix} = F\begin{pmatrix}x_0\\y_0\end{pmatrix} + \begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix} + E(h)$ with $\frac{\|E(h)\|}{\|h\|} \to 0$ as $h \to 0$

then

$u(z_0+h) + iv(z_0+h) = u(z_0) + iv(z_0) + (a+bi)(h_1+ih_2) + e(h)$, $\quad \left|\frac{e(h)}{h}\right| = \frac{\|E(h)\|}{\|h\|} \to 0$ as $h \to 0$

$f(z_0+h) = f(z_0) + (a+bi)h + e(h)$

$\frac{f(z_0+h) - f(z_0)}{h} = a+bi + \frac{e(h)}{h}$ ∎