Iteration of Rational Functions

by

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I. Newton's Method

P a polynomial of degree n, Newton's Method (Newton-Raphson method) for finding the roots of

$$P(z) = 0,$$

iteratively, is to study the iteration scheme

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}.$$

$$= N(z_n).$$

We compute

$$N'(z) = 1 - \frac{(P'(z))^2 - P''(z)P(z)}{(P'(z))^2}$$

$$= \frac{P''(z)P(z)}{(P'(z))^2}.$$

Hence if z^* is a simple root of N, we have that

$$N'(z^*) = 0.$$

On the other hand if z^* is a root of multipl k, we take as our iteration scheme

$$N_h(z) = z - \frac{hP(z)}{P'(z)}.$$

We then get

$$N_h'(z) = 1 - h\,\frac{P'(z)^2 - P(z)P''(z)}{[P'(z)]^2}.$$

Since

$$P(z) = (z-z^*)^k Q(z), \quad Q(z^*) \neq 0$$

we obtain

$$P'(z) = k(z-z^*)^{k-1}Q(z) + (z-z^*)^k Q'(z$$

$$P''(z) = k(k-1)(z-z^*)^{k-2}Q(z) + 2k(z-z^*)^{k-1}Q'(z) +$$

$$+ (z-z^*)^k \, Q''(z).$$

Hence

$$N_h'(z) = 1 - h + h \, \frac{(z-z^*)^{2k-2}\left[k(k-1)Q^2(z) + 2k(z-z^*)QQ' + (z-z^*)^2 Q''Q\right]}{(z-z^*)^{2k-2}\left[k\,Q(z) + (z-z^*)\,Q'\right]^2}$$

hence

$$N_h'(z) \longrightarrow 1 - h + \frac{h\,Q^2(z^*)\,k(k-1)}{k^2\,Q^2(z^*)}$$

$$= 1 - h + \frac{h\,k(k-1)}{k^2}$$

$$= 1 - h + h\,\frac{(k-1)}{k}$$

$$= 1 - \frac{h}{k}.$$

Hence, if we choose $h = k$, we again get

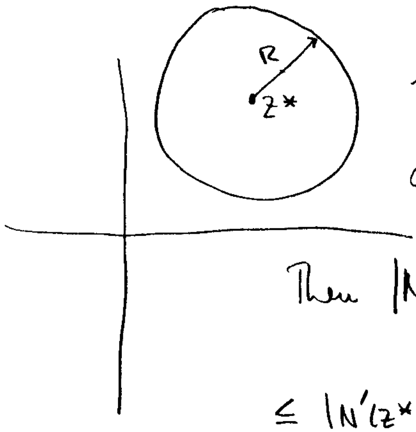
$$N_h'(z^*) = 0.$$

(3) Take any iteration scheme

$$z_{n+1} = N(z_n),$$

where

$$N(z^*) = z^*, \quad N'(z^*) = 0$$



let $M = \sup_{|z-z^*| \le R} |N''(z)|$

and let $|z - z^*| \le \alpha R$.

Then $|N(z) - N(z^*)| = |N(z) - z^*|$

$$\le |N'(z^*)(z-z^*)| + \left|\frac{N''(z)}{2!}\right| |z-z^*|^2$$

$$\le \frac{M}{2!}\alpha^2 R^2 \le \alpha R$$

provided

$$\frac{M}{2!}\alpha R \le 1$$

i.e.

$$\alpha \le \frac{2}{MR}.$$

Hence, if we choose α this way, then if we choose

z_0 so that

$$|z_0 - z^*| \le \alpha R,$$

then the iteration scheme

$$z_{n+1} = N(z_n),$$

will satisfy that $\{z_n\} \in \{z : |z - z^*| \le \alpha R\}$.

Also, since

$$|N(z) - N(\tilde{z})| = |N'(\bar{\tilde{z}})|\,|z - \bar{z}|,$$

it follows further that if we choose α so small that

$$|N'(\bar{\tilde{z}})| \le k < 1,$$

then N is a contraction mapping and hence has a unique fixed point, i.e., the iteration scheme will converge to z^*, and z^* is an attractive fixed point, i.e., there exists a neighborhood of z^* such that all points in the neighborhood will converge to z^* upon iteration.

II. Cayley - Schröder Problem.

Given an iteration scheme

$$z_{n+1} = N(z_n),$$

where N is a rational function as above and assume

$$N(z^*) = z^*,$$

where z^* is an attractive fixed point (i.e., $|N'(z^*)| = \lambda < 1$). Consider the set

$$A(z^*) = \{z : N^n(z) \longrightarrow z^*,\ n = 1, 2, \cdots\}$$

the basin of attraction of z^*. What is $A(z^*) = ?$.

Example $P(z) = z^2 - 1$.

Solutions

$$z^* = \pm 1.$$

The Newton method

is defined by

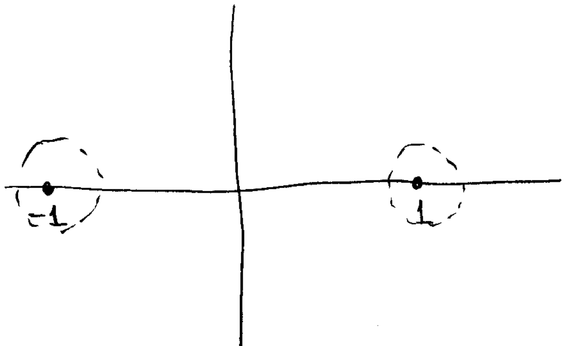
$$z_{n+1} = z_n - \frac{z_n^2 - 1}{2z_n}$$

$$= \frac{2z_n^2 - z_n^2 + 1}{2z_n}$$

$$= \frac{z_n^2 + 1}{2z_n}$$

$$= \frac{1}{2}\left(z_n + \frac{1}{z_n}\right).$$

Both $z^* = 1$ and $z^* = -1$ are attractive fixed points.



Conjecture: $RHP = A(1)$, $LHP = A(-1)$.

Note: If $\operatorname{Re} z_0 = 0$, then $\operatorname{Re}(N(z_0)) = 0$.

$$N(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{\bar{z}}{|z|^2}\right)$$

$$\operatorname{Re} N(z) = \frac{1}{2}\operatorname{Re}(z) + \frac{1}{2}\operatorname{Re}\left(\frac{\bar{z}}{|z|^2}\right)$$

$$= 0.$$

Thus $i\mathbb{R} \cap A(1) = \phi = i\mathbb{R} \cap A(-1)$.

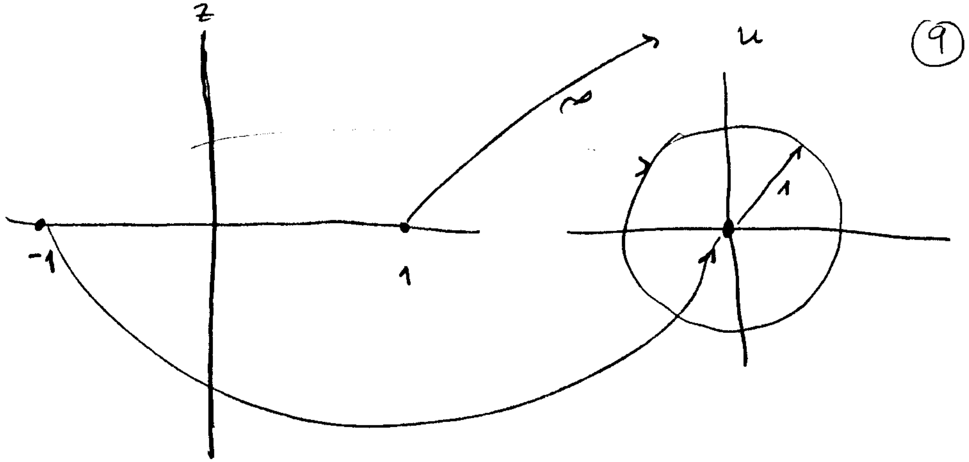
Also $N(i\mathbb{R}) \subset i\mathbb{R}$.

We verify the conjecture.

Let us consider the following Möbius transformation

$$u = \frac{z+1}{z-1}$$

(9)



$\operatorname{Re} z = 0$; $u = \dfrac{z+1}{z-1}$; $|u| = \dfrac{|(\operatorname{Im} z) i + 1|}{|(-\operatorname{Im} z) i + 1|}$

$= 1$

$u(z-1) = z+1$; $z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right)$

$z(u-1) = 1+u$; $= \frac{1}{2}\left(\frac{u_n+1}{u_n-1} + \frac{u_n-1}{u_n+1}\right)$

$z = \dfrac{u+1}{u-1}$

$$\frac{u_{n+1}+1}{u_{n+1}-1} = \frac{1}{2}\,\frac{(u_n+1)^2 + (u_n-1)^2}{u_n^2 - 1}$$

$$= \frac{1}{2}\,\frac{2u_n^2 + 2}{u_n^2 - 1}$$

$$= \frac{u_n^2+1}{}$$

(10)

Hence

$$u_{n+1} = u_n^2 .$$

Note that the above may also be written as follows

$$R = \phi \circ N \circ \phi^{-1},$$

where

$$\phi^{-1}(u) = \frac{u+1}{u-1}$$

and

$$R(u) = u^2$$

or

$$N = \phi^{-1} \circ R \circ \phi$$

We next study the mapping R.

1) $|u| = 1 \Rightarrow |R^n(u)| = 1$, $n = 1, 2, \dots$

2) $|u| > 1 \Rightarrow |R^n(u)| \longrightarrow \infty$, $n \to \infty$

3) $|u| < 1 \Rightarrow |R^n(u)| \longrightarrow 0$, $n \to \infty$

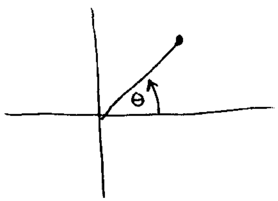
1) $\operatorname{Re} z = 0 \Rightarrow \operatorname{Re} N^n(z) = 0$, $n = 1, 2, \dots$

2) $\operatorname{Re} z > 0 \Rightarrow N^n(z) \longrightarrow 1$, $n \to \infty$

3) $\operatorname{Re} z < 0 \Rightarrow N^n(z) \longrightarrow -1$, $n \to \infty$.

The map $u \mapsto u^2$, $|u| = 1$.

$$u = e^{i\theta} = e^{i(2\pi\alpha)} = e^{2\pi i(\alpha)}$$



$$u^2 = e^{2\pi i(2\alpha)}$$

hence

$$u \longmapsto u^2$$

is

$$\alpha \underset{r(\alpha)}{\longmapsto} 2\alpha \pmod 1$$

// Binary expansions

$$\alpha = \sum_{k=1} \alpha_k 2^{-k}, \quad \alpha_k \in \{0, 1\}.$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k, \dots)$$

$$2\alpha \bmod 1 = (\alpha_2, \alpha_3, \dots) \quad \text{(Shift operator)}$$

α a fixed point if $\alpha = (0, 0, \dots, 0)$
$\alpha = (1, 1, \dots) = (0, 0, \dots, 0, \dots)$

α a point of period 2 if

$$\alpha = (\alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \dots)$$

- There are periodic points of any period !!

- Given any α, any $\varepsilon > 0$, there always exists $\delta > 0$, β and n, with $\operatorname{dist}(\alpha, \beta) \le \varepsilon$ and $\operatorname{dist}(r^n(\alpha), r^n(\beta)) \ge \delta$.

$$\alpha = \sum_{k=1}^{\infty} \alpha_k 2^{-k}$$

$$\beta = \sum_{k=1} \beta_k 2^{-k}$$

$$\operatorname{dist}(\alpha, \beta) \le \sum_{k=1}^{\infty} |\alpha_k - \beta_k| 2^{-k}$$

Hence if $\alpha_i = \beta_i$, $i = 1, \dots, N$

$$\mathrm{dist}(\alpha, \beta) \leq \sum_{k=N+1}^{\infty} 2\, 2^{-k}$$

On the other hand, if $\alpha_j \neq \beta_j$ $j = N+1, \dots$

then

$$\mathrm{dist}(r^N(\alpha), r^N(\beta)) \geq \sum_{1}^{\infty} 2^{-k}.$$

- Given any α and $\varepsilon > 0$, there exists a periodic point β with $\mathrm{dist}(\alpha, \beta) < \varepsilon$.

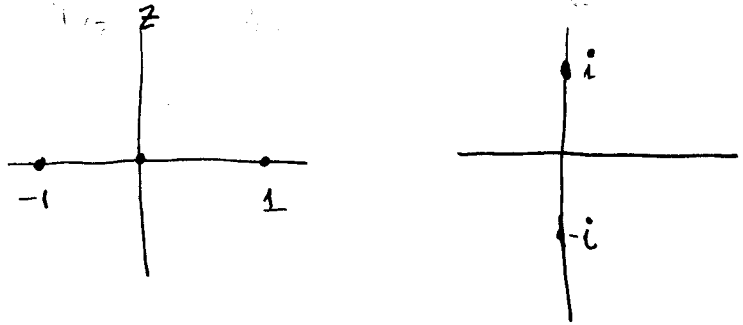
- There are numbers α such that $r^n(\alpha)$ comes arbitrarily close to any β.
  (i.e. given $\varepsilon > 0$, there exists n such that $\mathrm{dist}(\beta, r^n(\alpha)) < \varepsilon$)
  Example:

$$\alpha = (1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, \underbrace{1, 0, 0, 0}_{4}, \dots$$

1
1 0 1 0
1 1 1 1 1 1
1 0 0 1 0 0 1 0 0
1 1 0 1 1 0 1 1 0 1 1 0

What about other quadratics

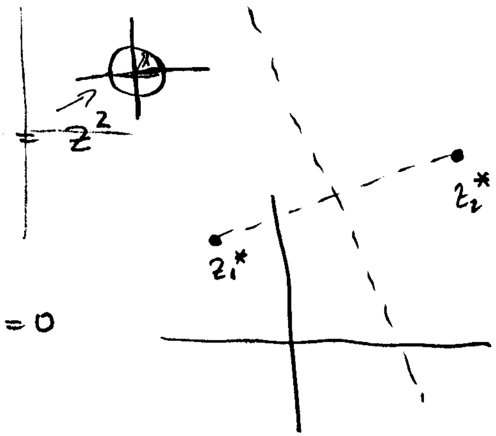
- $P(z) = z^2 + 1$ $\quad u =$



- $P(z) = z^2 + az + b$

$$= (z + \tfrac{a}{2})^2 + b - \tfrac{a^2}{4}$$

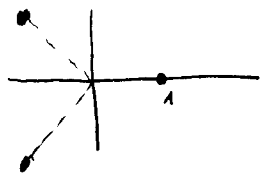
- $P(z) = z^2$



- $P(z) = 0$

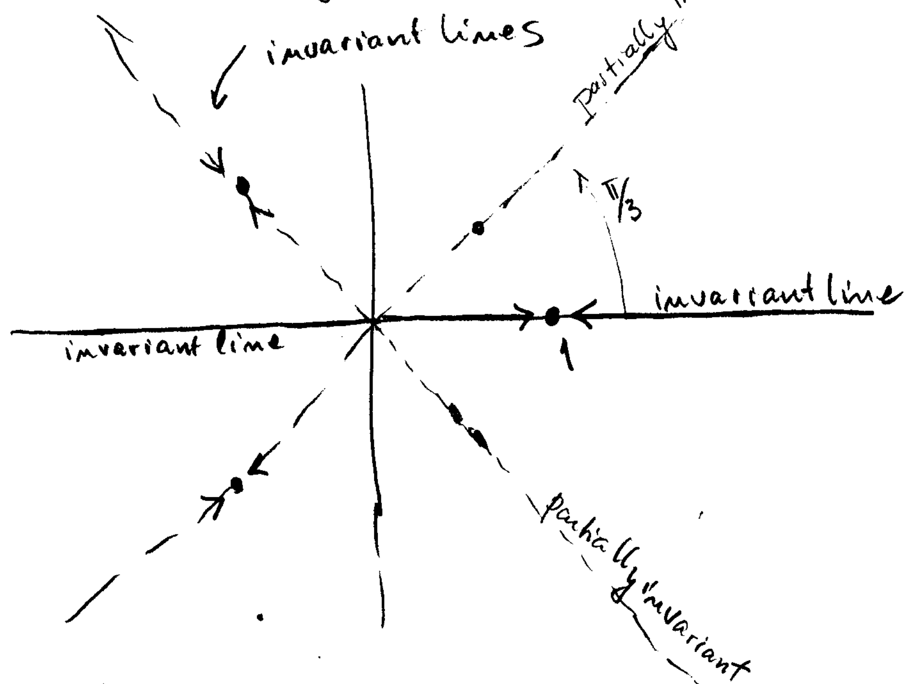
Other polynomials

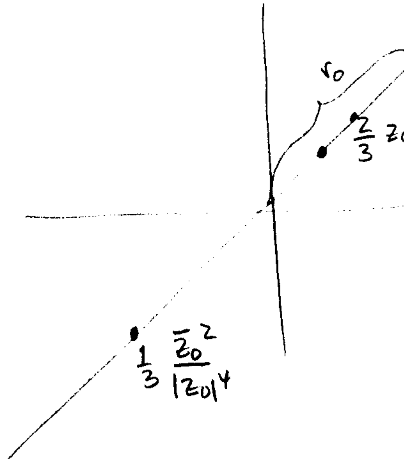
$$P(z) = z^3 - 1$$



$$N(z) = z - \frac{z^3-1}{3z^2} = \frac{3z^3 - z^3 + 1}{3z^2} = \frac{2}{3}z + \frac{1}{3z^2}.$$

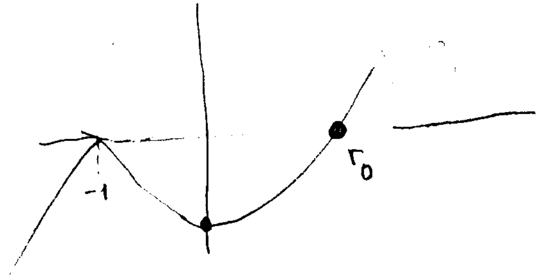
$$\arg N(z) = \arg\left(\frac{2z^3+1}{3z^2}\right)$$





$$-\frac{2}{3}r + \frac{1}{3r^2} = 1$$

$$3r^2 + 2r^3 - 1 = 0$$



If $|z_0| = r$ is such that $\frac{1}{3r^2} < \frac{2}{3}r$
$\arg z_0 = \frac{\pi}{3}$

i.e. $r^3 > \frac{1}{2}$

then $\arg N(z_0) = \pi/3$

whereas if $\frac{1}{3r^2} > \frac{2}{3}r$, then $\arg N(z_0) = -\frac{\pi}{3}$

If z_0 is such that $\frac{1}{3r^2} = \frac{2}{3}r$, then $N(z_0) = 0$
and $z_0 \to 0 \to \infty$

Work of Julia + Fatou

P, Q polynomials

$$R = \frac{P}{Q}$$

rational function

$$\deg R = \max\{\deg P, \deg Q\} > 1.$$

Iteration scheme

$$z_{n+1} = R(z_n)$$

- $O^+(z_0) = \{z_0, R(z_0), R^2(z_0), \dots\}$

- If for some n, $R^n(z_0) = z_0$, z_0 is a periodic point. The least such n is called the period of z_0.

- If z_0 a point of period n, the number

$$\lambda = (R^n)'(z_0)$$

is the eigenvalue of z_0.

- z_0 is
  - superattractive $\Leftrightarrow \lambda = 0$
  - attractive $\Leftrightarrow 0 < |\lambda| < 1$
  - indifferent $\Leftrightarrow |\lambda| = 1$
  - repelling $\Leftrightarrow |\lambda| > 1$.

- The Julia set J_R of R:

$$P = \{z_0 : z_0 \text{ is periodic and repelling}\}$$

$J_R = \overline{P}$ (each point in J_R is the limit of a sequence of points from P).

- If z_0 a fixed point

$$A(z_0) = \{z : R^n(z) \to z_0\},$$

the basin of attraction of z_0. (Note $A(z_0) \neq \phi$ if z_0 is an attractive or superattractive fixed point).

Basic Result:

R rational function, $\deg R > 1$. Then:

(i) $J_R \neq \phi$, J_R is an uncountable set.

(ii) $J_R = J_{R^k}$, $k = 1, 2, \dots$

(iii) $R(J_R) = J_R = R^{-1}(J_R)$.

(iv) For any $z \in J_R$, $O^-(z) = \bigcup_{k=1}^{\infty} \{w : R^k(w) = z\}$ is dense in J_R

(V) If $\gamma = O^+(z_0)$, where z_0 is an attractive periodic point, then $A(\gamma) \subset \overline{\mathbb{C}} \setminus J_R = F_R$ and

$$\partial A(\gamma) = J_R .$$

(VI) If J_R has interior points, then $J_R = \overline{\mathbb{C}}$ (example $R(z) = \dfrac{(z-2)^2}{z^2}$)

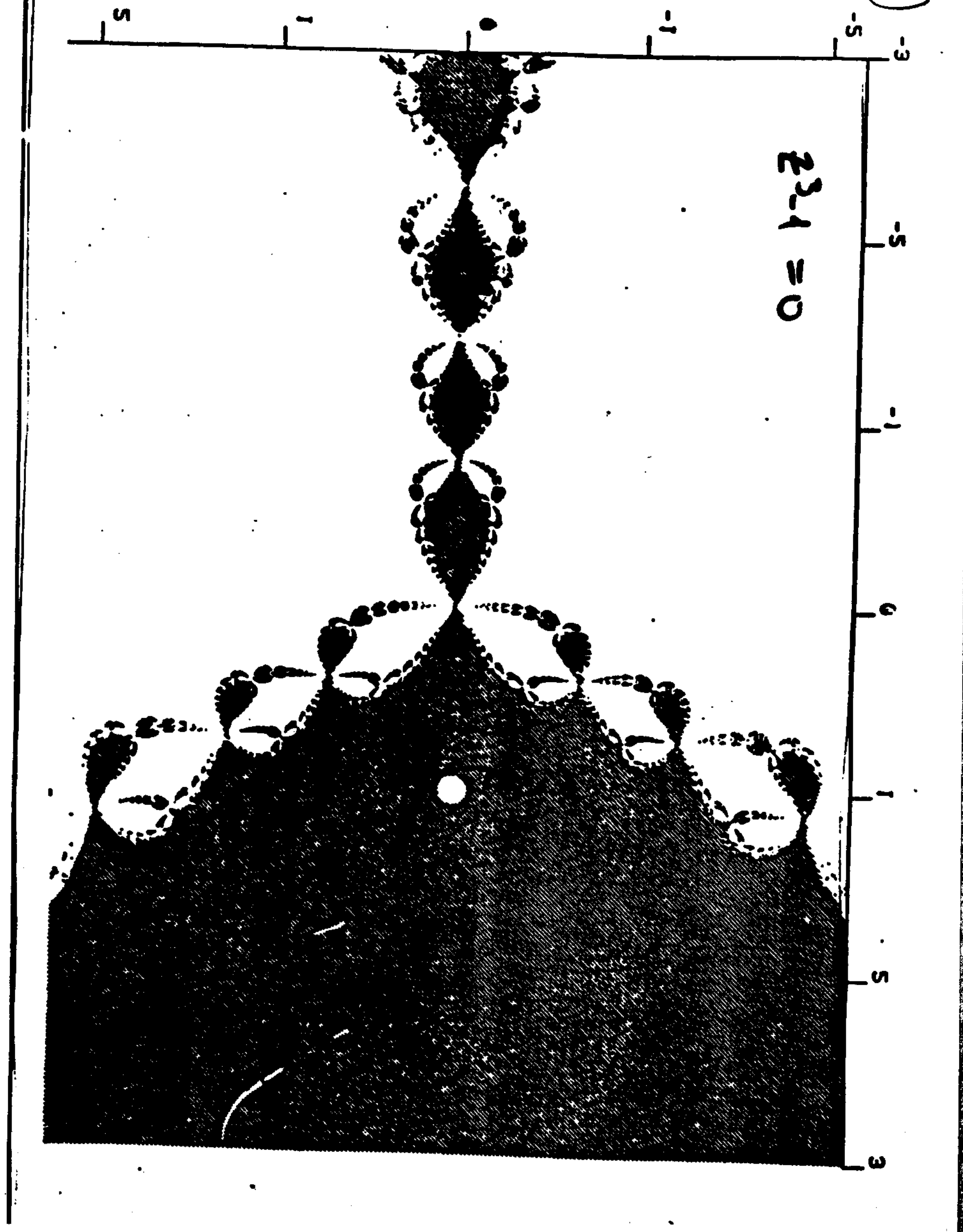
Example:
$$R(z) = \frac{2z^3+1}{3z^2}$$

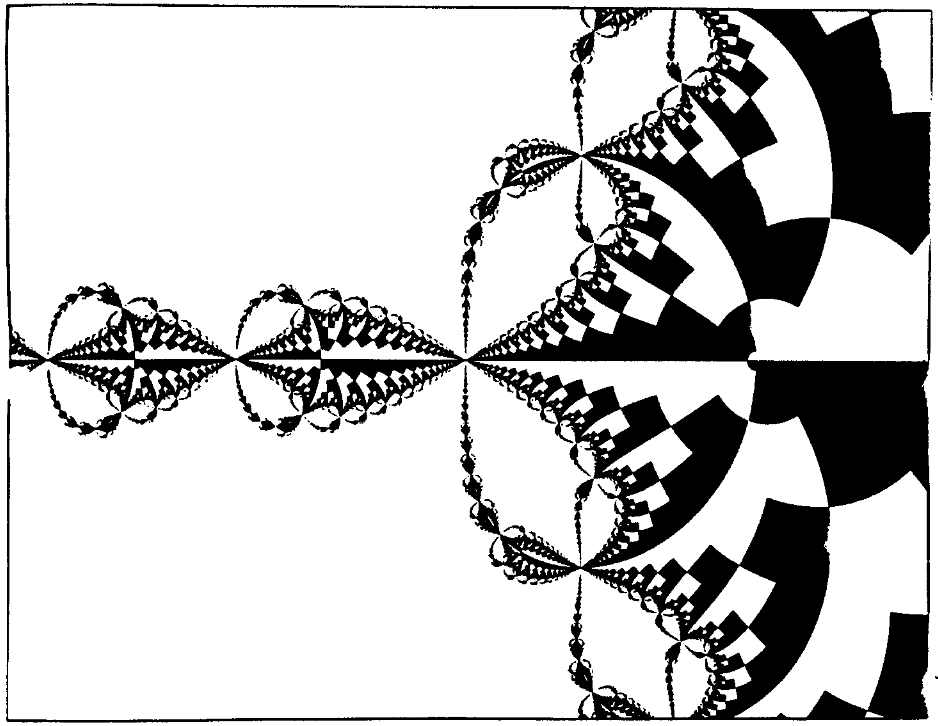
has three attractive fixed points

$$1, \quad e^{\frac{2\pi i}{3}}, \quad e^{\frac{4\pi i}{3}}$$

Hence

$$\partial A(1) = \partial A(e^{\frac{2\pi i}{3}}) = \partial A(e^{\frac{4\pi i}{3}}) .$$





The Mandelbrot Set

If $R(z) = az^2+bz+d$, $a \neq 0$

then by a change of coordinates we may assume that

$$R(z) = z^2 + c, \quad c \in \mathbb{C}.$$

Question: What is the Julia set of R.

Note that

$$R(\infty) = \infty$$

and ∞ is an attractive fixed point.

Hence

$$J_R = \partial A(\infty).$$

Also J_R is either connected or a Cantor set.

- $M = \{ c : J_R \text{ is connected} \}$.
- Julia + Fatou: J_R is connected if and only if $0 \notin A(\infty)$
- $c \in M$ if and only if $0 \to c \to c^2+c \to \cdots$

is a bounded sequence.

- M is a connected set (Douady/Hubbard)
- $M \subset \{c \in \mathbb{C} : |c| \leq 2\}$ (v. Haeseler)

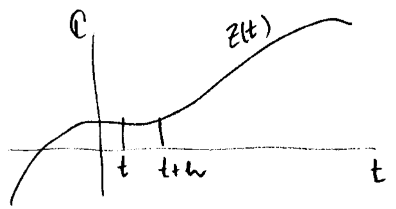
$(0 \to -2 \to 4-2 = 2 \to 4-2 \to 2)$

The Continuous Newton Method,

- Relaxed Newton method

$$z_{n+1} = z_n - h \frac{P(z_n)}{P'(z_n)}, \quad h \text{ a parameter } |h| < 1.$$

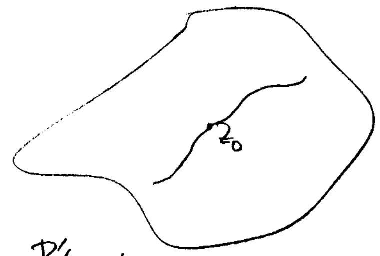
- $$\frac{z_{n+1} - z_n}{h} = - \frac{P(z_n)}{P'(z_n)}$$



$$\frac{z(t+h) - z(t)}{h} \approx z'(t)$$

- Continuous Newton Method

$$(\text{IVP}) \begin{cases} z'(t) = - \dfrac{P(z(t))}{P'(z(t))} \\ z(0) = z_0 \end{cases}$$



$P'(z_0) \neq 0$

- For each z_0 such that $P'(z_0) \neq 0$, there exists an interval $(t^-_{z_0}, t^+_{z_0})$ and a unique solution $z(t)$ of (IVP) defined on this interval. If $t^+_{z_0} < \infty$, then $z(t) \to \bar{z}$, where $P'(\bar{z}) = 0$.
- Solution satisfies

$$\frac{d}{dt} P(z(t)) = - P(z(t)).$$

Hence by a change of variables

$$P(z(t)) = P(z_0) e^{-t}.$$

- If $t_{z_0}^{+} = +\infty$, $P(z(t)) \longrightarrow 0$ and hence $z(t) \longrightarrow z^*$, where $P(z^*) = 0$.

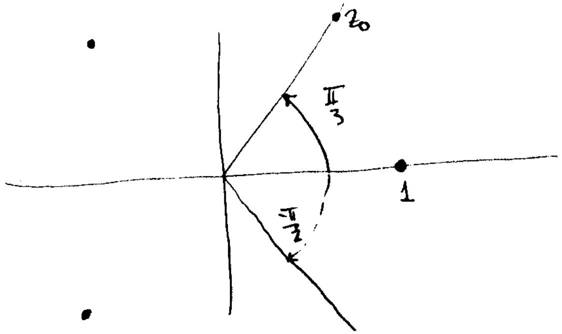
- Question: If z^* is such that $P(z^*) = 0$, what is $\{ z_0 : z(t) \longrightarrow z^* \}$, where $z(t)$ was defined above.

- Example. Consider $P(z) = z^3 - 1$, then for z_0 such that $P'(z_0) \neq 0$, we obtain

$$(*) \qquad z^3(t) - 1 = (z_0^3 - 1)e^{-t}.$$

Note that

$$\{ z_0 : P'(z_0) = 0 \} = \{0\}.$$



Note $z^3(t) = 0$ iff

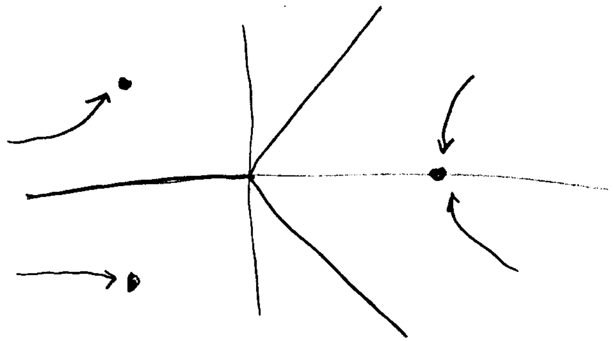
$$z_0^3 - 1 = -e^t$$

or

$$z_0^3 = 1 - e^t,$$

which is possible only for z_0 with $\arg z_0 = \frac{\pi}{3}$, $\arg z_0 = -\frac{\pi}{3}$ and $\arg z_0 = \pi$.

We hence obtain the phase diagram



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