Math 4530
Monday 25 March

Multi-variable FTC

$\boxed{\text{1st recall 1-variable version:}}$

$$\int_a^b f(x)\,dx = F(b)-F(a), \quad \text{where } F'(x)=f(x)$$

or

$$\boxed{\int_a^b F'(x)\,dx = F(b)-F(a)}$$

A good way to think of this:

$$F'(x) \approx \frac{F(x+h)-F(x)}{h}$$

so

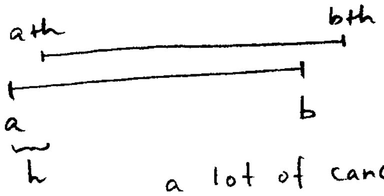
$$\int_a^b F'(x) \overset{①}{\approx} \int_a^b \frac{F(x+h)-F(x)}{h}\,dx$$

$$= \frac{1}{h}\int_a^b F(x+h)\,dx - \frac{1}{h}\int_a^b F(x)\,dx$$

$z = x+h$
$dz = dx$

(translation of variables)

$$= \frac{1}{h}\int_{a+h}^{b+h} F(z)\,dz - \frac{1}{h}\int_a^b F(x)\,dx$$



a lot of cancellation!

$$= \frac{1}{h}\int_b^{b+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx$$

$$\overset{②}{\approx} F(b)-F(a)$$

In fact, as $h \to 0$ you can deduce

$$\int_a^b F'(x) = F(b)-F(a)$$

[For first $\approx$ you need e.g. $F'(x)$ continuous on $[a,b]$
For second $\approx$ you need e.g. $F(x)$ continuous at a & b]

Details:

$\overset{①}{\approx}$: By MVT

$$\frac{F(x+h)-F(x)}{h} = F'(c_x), \quad |x-c_x|<h$$

so

$$\left|\int_a^b F'(x) - \left(\frac{F(x+h)-F(x)}{h}\right)dx\right| \le \int_a^b |F'(x)-F'(c_x)|\,dx,$$

$|x-c_x|<h$

If F' unif cont on $[a,b]$
then $\forall \varepsilon>0\ \exists\, \delta>0$ s.t.
$|x-c_x|<\delta$
$\Rightarrow |F'(x)-F'(c_x)|<\varepsilon$

in which case,

$$\le \int_a^b \varepsilon\,dx = \varepsilon(b-a) \quad \blacksquare$$

$\overset{②}{\approx}$:

$$\left|\frac{1}{h}\int_b^{b+h} F(x)\,dx - F(b)\right| = \left|\frac{1}{h}\int_b^{b+h} F(x)-F(b)\,dx\right|$$

F cont at $b \Rightarrow \forall \varepsilon>0\ \exists\, \delta>0$
s.t. $|F(x)-F(b)|<\varepsilon$
when $|x-b|<\delta$

in which case

$$\le \frac{1}{h}\int_b^{b+h} \varepsilon\,dx = \varepsilon \quad \blacksquare$$

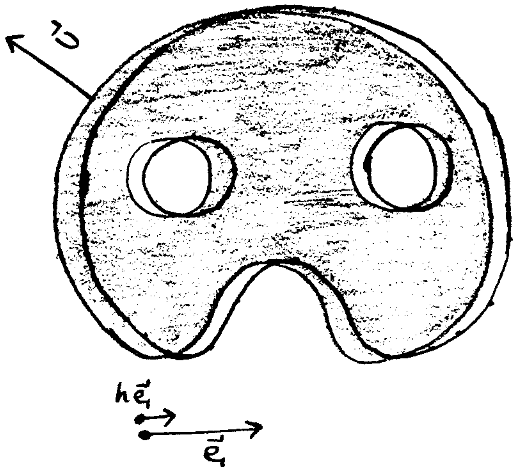
The same idea is behind $\mathbb{R}^n$ version
(although proof usually follows a technically simpler route)

Illustrate when $n=2$

Domain D (shaded), ∂D = bdry curve

$D_h := \{ z = x + h\vec{e}_1 \text{ s.t. } x \in D \}$

is translation by amt h in $\vec{e}_1$ direction



$\vec{\nu}$ = unit exterior normal

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$F: \mathbb{R}^2 \to \mathbb{R}$

$$\iint_D \frac{\partial F}{\partial x}\, dA \overset{①}{\approx} \iint_D \frac{F(\vec{x}+h\vec{e}_1) - F(\vec{x})}{h}\, dA$$

$$= \frac{1}{h}\iint_D F(\vec{x}+h\vec{e}_1)\, dA \quad - \frac{1}{h}\iint_D F(\vec{x})\, dA$$

$\vec{z} = \vec{x} + h\vec{e}_1$

$dA_z = dA_x$

(translation of domains)

$$= \frac{1}{h}\iint_{D_h} F(\vec{z})\, dA_z - \frac{1}{h}\iint_D F(\vec{x})\, dA_x$$

$\overset{①}{\approx}$ and $\overset{②}{\approx}$ are messy to prove, so rigorous proof is exposited slightly differently in general (page 4), but with this idea hiding behind it.

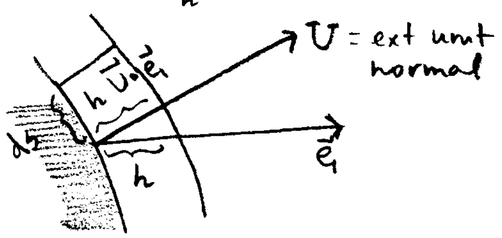
lots of cancellation in the middle

$S_h^+ = D_h \setminus D$ = unshaded strip(s)

$S_h^- = D \setminus D_h$ = shaded strip(s)

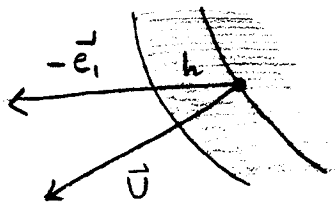
$$* \quad = \frac{1}{h}\iint_{S_h^+} F(x)\, dA \quad - \frac{1}{h}\iint_{S_h^-} F(x)\, dA$$

near S_h^+



$dA \approx ds\, h(\vec{\nu}\cdot\vec{e}_1)$

near S_h^-



$dA \approx h\, ds\, (\vec{\nu}\cdot(-\vec{e}_1))$

So $\displaystyle * \overset{②}{\approx} \int_{\partial D} F\vec{e}_1\cdot\vec{\nu}\, ds$ (h's cancel, also signs in 2nd term)

In the limit $\boxed{\displaystyle \iint_D \frac{\partial F}{\partial x}\, dA = \int_{\partial D} F\vec{e}_1\cdot\vec{\nu}\, ds}$

$\mathbb{R}^1$	$\mathbb{R}^n$
$\int_a^b F'(x)\,dx = F(b) - F(a)$	$\int_{D_n} \frac{\partial F}{\partial x_i}\, dVol_n = \int_{\partial D_n} F(\vec{e}_i \cdot \vec{U})\, dVol_{n-1}$

SAME!

Special forms: $\boxed{\text{Divergence Theorem}}$

$$\operatorname{div}\begin{bmatrix} P \\ Q \end{bmatrix} := P_x + Q_y$$

$$\iint_D P_x \, dA = \int_{\partial D} P\, \vec{e}_1 \cdot \vec{U}\, ds$$

$$+ \quad \iint_D Q_y \, dA = \int_{\partial D} Q\, \vec{e}_2 \cdot \vec{U}\, ds$$

$$\iint_D \operatorname{div}\begin{bmatrix} P \\ Q \end{bmatrix} dA = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \vec{U}\, ds$$

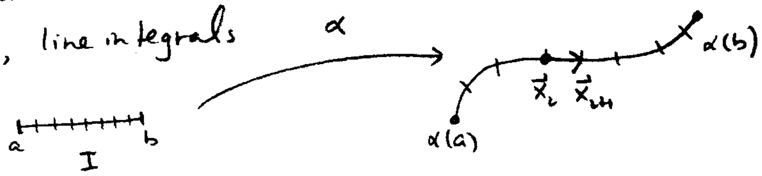
$\vec{F}(\vec{x})$ a vector field

in $\mathbb{R}^n$

$$\boxed{\int_{D_n} \operatorname{div}\vec{F}\, dVol_n = \int_{\partial D_n} \vec{F} \cdot \vec{U}\, dVol_{n-1}}$$

$\boxed{\text{Green's thm \& Stoke's Thm in } \mathbb{R}^2}$

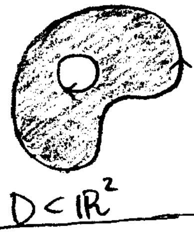
1^{st} recall, line integrals



$$\int_\alpha \vec{F} \cdot d\vec{x} := \lim_{\|P\| \to 0} \sum \vec{F}(\vec{x}_i) \cdot (\vec{x}_{i+1} - \vec{x}_i)$$

$$\vec{F}(\vec{\alpha}(t_i)) \cdot (\vec{\alpha}(t_{i+1}) - \vec{\alpha}(t_i))$$

$$= \int_I \vec{F}(\vec{\alpha}(t)) \cdot \vec{\alpha}'(t)\, dt$$



$D < \mathbb{R}^2$

depends on direction image curve is traversed, but not on particular parameterization. Line integrals can compute work

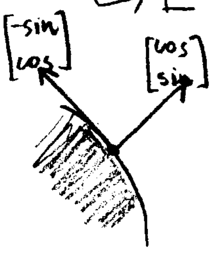
$$\boxed{\text{Green}: \iint_D Q_x - P_y \, dA = \oint_{\partial D} P\,dx + Q\,dy}$$

$$\boxed{\text{Stokes} \quad M^2 < \mathbb{R}^3 \quad \iint_M (\nabla \times \vec{F}) \cdot \vec{U}\, dA = \int_{\partial M} \vec{F} \cdot d\vec{x}}$$

(parametric version of Green's)

pf $\iint_D \operatorname{div}\begin{bmatrix} Q \\ -P \end{bmatrix} dA = \int_{\partial D} \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} \vec{U} \end{bmatrix} ds$

by div thm.



Now parameterize ∂D by arclength

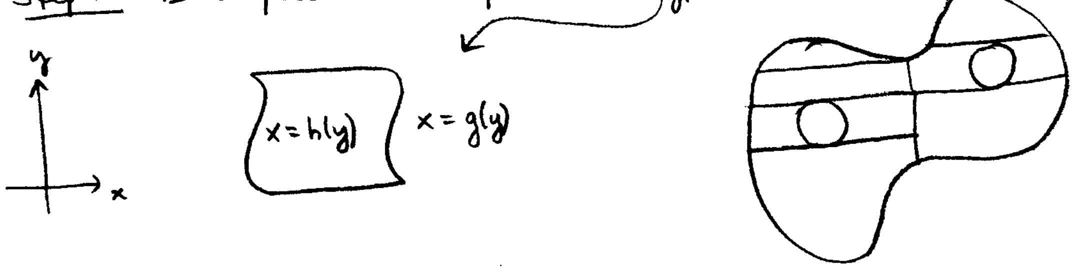
$$\text{RHS} = \int_{\partial D} \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} \cos \\ \sin \end{bmatrix} ds$$

$$= \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} -\sin \\ \cos \end{bmatrix} ds \text{ "=" } \oint P\,dx + Q\,dy$$

The way towards a rigorous proof of n-variable FTC
(illustrated for $n=2$)

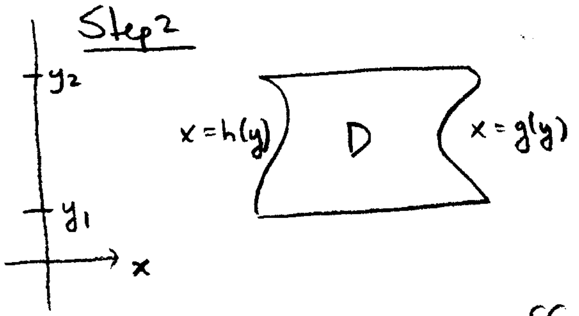
to show $$\iint_D \frac{\partial F}{\partial x}\,dA = \int_{\partial D} F\vec{e}_1\cdot\vec{U}\,ds$$

Step 1 Decompose D into pieces of type



if can show for basic piece,
add over all pieces, notice interior edge contributions from adjacent pieces cancel, so get thm.

Step 2



Fubini

$$\iint_D \frac{\partial F}{\partial x}\,dA = \iint_D \frac{\partial F}{\partial x}\,dx\,dy = \int_{y_1}^{y_2}\left(\int_{h(y)}^{g(y)} \frac{\partial F}{\partial x}\,dx\right)dy$$

$$= \int_{y_1}^{y_2} F(y,g(y)) - F(y,h(y))\,dy$$ 1-var FTC

$$= \int_{\gamma_1} F\,dy - \int_{\gamma_2} F\,dy$$

but on γ_1, γ_2, $ds = \sqrt{1+g'^2}\,dy$; $dy = \frac{1}{\sqrt{1+g'(y)^2}}\,ds$

$$= \int_{\partial D} F(\vec{e}_1\cdot\vec{U})\,ds$$ ■