

Math 2280-002

Week 9, March 4-8 5.1-5.3, introduction to Chapter 6

Mon Mar 4

5.1-5.2 Systems of differential equations - summary so far; converting differential equations (or systems) into equivalent first order systems of DEs; example with pplane visualization and demo.

Announcements: w 8.3 is postponed until next assignment
(What to do with complex eigendata when trying
to solve $\vec{x}'(t) = A\vec{x}$)

Warm-up Exercise: look over first few pages of "review". Any questions on
that material?

Summary of Chapters 4-5 so far (for reference):

Theorem 1 For the IVP

$$\begin{aligned}\mathbf{x}'(t) &= \mathbf{F}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

If $\mathbf{F}(t, \mathbf{x})$ is continuous in the t -variable and differentiable in its \mathbf{x} variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\begin{aligned}\mathbf{x}'(t) + P(t)\mathbf{x}(t) &= \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

If the matrix $P(t)$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval I containing t_0 then a solution $\mathbf{x}(t)$ exists and is unique, on the entire interval.

Theorem 3 Vector space theory for first order systems of linear DEs (We noticed the familiar themes... we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\mathbf{x}(t)$ differentiable on an interval, the operator

$$L(\mathbf{x}(t)) := \mathbf{x}'(t) + P(t)\mathbf{x}(t)$$

is linear, i.e.

$$\begin{aligned}L(\mathbf{x}(t) + \mathbf{z}(t)) &= L(\mathbf{x}(t)) + L(\mathbf{z}(t)) \\ L(c\mathbf{x}(t)) &= cL(\mathbf{x}(t)).\end{aligned}$$

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\mathbf{x}'(t) + P(t)\mathbf{x}(t) = \mathbf{f}(t)$$

$\forall t \in I$ is

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_H(t)$$

where $\mathbf{x}_p(t)$ is any single particular solution and $\mathbf{x}_H(t)$ is the general solution to the homogeneous problem

$$\mathbf{x}'(t) + P(t)\mathbf{x}(t) = \mathbf{0}.$$

3.3) (Generalizes what we talked about on Friday last week.) For $P(t)_{n \times n}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\mathbf{x}'(t) + P(t)\mathbf{x}(t) = \mathbf{0}$$

is n -dimensional. Here's why:

- Let $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$ defined by

$$[W(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)](t_0) := [\mathbf{X}_1(t_0) | \mathbf{X}_2(t_0) | \dots | \mathbf{X}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible Wronskian matrix.)

- Then for any $\mathbf{b} \in \mathbb{R}^n$ the IVP

$$\begin{aligned} \mathbf{x}'(t) + P(t)\mathbf{x}(t) &= \mathbf{0} \\ \mathbf{x}(t_0) &= \mathbf{b} \end{aligned}$$

has solution $\mathbf{x}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \dots + c_n\mathbf{X}_n(t)$ where the linear combination coefficients comprise the solution vector to the Wronskian matrix equation

to solve IVP
at t_0 , solve
this

$$\begin{bmatrix} | & | & | & | \\ \mathbf{X}_1(t_0) & \mathbf{X}_2(t_0) & \dots & \mathbf{X}_n(t_0) \\ | & | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$, and since each IVP has only one solution, the vector functions $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\mathbf{b} = \mathbf{0}$) is the one with $\mathbf{c} = \mathbf{0}$. Thus $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and the number n is the dimension of the solution space.

New today: Along with the sum rule and constant scalar multiple rule for vector (or matrix) valued functions, we will be using the Universal product rule: Shortcut to take the derivatives of

- $f(t)\mathbf{x}(t)$ (scalar function times vector function),
- $f(t)A(t)$ (scalar function times matrix function),
- $A(t)\mathbf{x}(t)$ (matrix function times vector function),
- $\mathbf{x}(t) \cdot \mathbf{y}(t)$ (vector function dot product with vector function),
- $\mathbf{x}(t) \times \mathbf{y}(t)$ (cross product of two vector functions),
- $A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of t , and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t) * B(t)) = A'(t) * B(t) + A(t) * B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. There is one explanation that proves all of those product rules at once:

example

$$\frac{d}{dt} \begin{bmatrix} t & \cos 3t \\ 17 & e^{10t} \end{bmatrix} \begin{bmatrix} 12t^2 + 7 \\ 6 \sin 4t \end{bmatrix} = \begin{bmatrix} 1 & -3 \sin 3t \\ 0 & 10e^{10t} \end{bmatrix} \begin{bmatrix} 12t^2 + 7 \\ 6 \sin 4t \end{bmatrix} + \begin{bmatrix} t & \cos 3t \\ 17 & e^{10t} \end{bmatrix} \begin{bmatrix} 24t \\ 24 \cos 4t \end{bmatrix}$$

$A' B \quad \quad \quad A B'$

$$\begin{aligned} \frac{d}{dt} (A * B)(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [A(t + \Delta t) * B(t + \Delta t) - A(t) * B(t)] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\underbrace{A(t + \Delta t) * B(t + \Delta t) - A(t) * B(t + \Delta t)}_{(A(t + \Delta t) - A(t)) * B(t + \Delta t)} + \underbrace{A(t) * B(t + \Delta t) - A(t) * B(t)}_{A(t) * (B(t + \Delta t) - B(t))} \right] \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{1}{\Delta t} (A(t + \Delta t) - A(t)) * B(t + \Delta t) + A(t) * \left(\frac{1}{\Delta t} (B(t + \Delta t) - B(t)) \right) \right] \end{aligned}$$

\downarrow
 $A'(t) * B(t)$

 \downarrow
 $A(t) * B'(t)$

limit of sums are sums of limits
 & limits of products are products of limits

Theorem 4) The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and the tank example): For the homogeneous system of differential equations

$$\mathbf{x}'(t) = A \mathbf{x}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$, $A_{n \times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of \mathbb{R}^n made out of eigenvectors of A , i.e. $A \mathbf{v}_j = \lambda_j \mathbf{v}_j$ for each $j = 1, 2, \dots, n$), then the functions

$$e^{\lambda_j t} \mathbf{v}_j, \quad j = 1, 2, \dots, n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\mathbf{x}_H(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + \dots + c_n e^{\lambda_n t} \mathbf{v}_n.$$

proof: check the Wronskian matrix at $t=0$, it's the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for \mathbb{R}^n (or \mathbb{C}^n).

if $\vec{X}_j(t) = e^{\lambda_j t} \vec{v}_j$ where $A \vec{v}_j = \lambda_j \vec{v}_j$

then $\vec{X}_j'(t) = \lambda_j e^{\lambda_j t} \vec{v}_j \quad (+ e^{\lambda_j t} \vec{0})$ ← LHS

$A \vec{X}_j(t) = A e^{\lambda_j t} \vec{v}_j = e^{\lambda_j t} A \vec{v}_j = e^{\lambda_j t} \lambda_j \vec{v}_j$ ← RHS

↑
scalar

Solve IVP's at $t=0$, get

at $t=0$ →

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

uniquely solvable.

Discussed on Tuesday 3/5

There is an alternate direct proof of Theorem 4 which is based on diagonalization from Math 2270. You are asked to use the idea of this alternate proof to solve a nonhomogeneous linear system of DE's this week - in homework problem w8.4e.

proof 2:

Definition: Let $A_{n \times n}$. If there is an \mathbb{R}^n (or \mathbb{C}^n) basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ consisting of eigenvectors of A , then A is called diagonalizable. This is precisely why:

Write $A \mathbf{v}_j = \lambda_j \mathbf{v}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the invertible matrix with those eigenvectors as columns:

$$P = [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n]$$

Then, using the various ways of understanding matrix multiplication, we see

$$\begin{aligned} AP &= A[\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 | \lambda_2 \mathbf{v}_2 | \dots | \lambda_n \mathbf{v}_n] \\ &= [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \end{aligned}$$

$$AP = PD$$

$$A = PD P^{-1}$$

$$P^{-1}AP = D$$

mult on right by P^{-1}
mult on left by P^{-1}

w8.4
 $\vec{f}(t) = \text{const vector}$
↓

Now let's feed this into our system of differential equations

$$\mathbf{x}'(t) = A \mathbf{x}$$

let's change functions in our DE system:

$$\mathbf{x}(t) = P \mathbf{u}(t), \quad (\mathbf{u}(t) = P^{-1} \mathbf{x}(t))$$

and work out what happens (and see what would happen if the system of DE's was non-homogeneous, as in your homework).

$$\begin{aligned} (P \mathbf{u}(t))' &= A P \mathbf{u}(t) \\ P \mathbf{u}'(t) &= A P \mathbf{u}(t) \\ P^{-1} \text{ on left } \mathbf{u}'(t) &= \underbrace{P^{-1} A P}_D \mathbf{u} \end{aligned}$$

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$\vec{x}'(t) = A \vec{x} + \vec{f}(t)$$

$$\vec{x}(t) = P \vec{u}(t)$$

$$\begin{aligned} (P \vec{u})' &= A P \vec{u} + \vec{f}(t) \\ P \vec{u}' &= A P \vec{u} + \vec{f}(t) \\ \vec{u}' &= \underbrace{P^{-1} A P}_D \vec{u} + \underbrace{P^{-1} \vec{f}}_{\vec{g}(t)} \end{aligned}$$

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 + g_1(t) \\ \lambda_2 u_2 + g_2(t) \\ \vdots \\ \lambda_n u_n + g_n(t) \end{bmatrix}$$

$$u_1' = \lambda_1 u_1$$

$$u_2' = \lambda_2 u_2$$

⋮

$$u_n' = \lambda_n u_n$$

So
Chptr 1

$$u_1(t) = c_1 e^{\lambda_1 t}$$

$$u_2(t) = c_2 e^{\lambda_2 t}$$

⋮

$$u_n(t) = c_n e^{\lambda_n t}$$

constants
in HW
(constant coeff 1st
order linear DE's
are our favorites -
Chapter 1 & Chapter 3)

So

$$\vec{x}(t) = P \vec{u}(t)$$

$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n$$

(same!)

New today: It is always the case that an initial value problem for one or more differential equations of arbitrary order is equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example. (See examples and homework problems in section 4.1) This gives us a new perspective on the way we solved differential equations from Chapter 3.

For example, consider this overdamped problem from Chapter 3:

$$\begin{aligned}x''(t) + 7x'(t) + 6x(t) &= 0 \\x(0) &= 1 \\x'(0) &= 4.\end{aligned}$$

Exercise 1a) Do enough checking to verify that $x(t) = 2e^{-t} - e^{-6t}$ is the solution to this IVP.

check

$$\begin{aligned}p(r) &= r^2 + 7r + 6 \\&= (r+6)(r+1) \\x_H(t) &= c_1 e^{-t} + c_2 e^{-6t} \checkmark\end{aligned}$$

$$\begin{aligned}x(0) &= 2 - 1 = 1 \\x'(0) &= -2 + 6 = 4 \checkmark\end{aligned}$$

1b) Show without using the formula for its solution, that if $x(t)$ solves the IVP above, then the vector function $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

IVP 2

$$\begin{cases} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{cases}$$

Then use the solution from 1a to write down the solution to the IVP in 1b.

So soln to IVP2 is

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

check

$$\begin{aligned}\frac{d}{dt} \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} &= \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix} \\ \text{but } x''(t) &= -6x - 7x' \\ &\downarrow \\ &= \begin{bmatrix} x' \\ -6x - 7x' \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} \\ \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}\end{aligned}$$

1c) Show that if $[x_1(t), x_2(t)]^T$ solves the IVP in 1b then without using the formula for the solution, show that the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure (when the system arises that way).

$$x_1' = x_2$$

$$x_2' = -6x_1 - 7x_2$$

$$x_1(0) = 1$$

$$x_2(0) = 4$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So $x_1'' = x_2'$

$$= -6x_1 - 7x_2$$

$$= -6x_1 - 7x_1'$$

So $x_1'' + 7x_1' + 6x_1 = 0$ ✓

$$x_1(0) = 1$$

$$x_1'(0) = x_2(0) = 4$$

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$x(0) = 1$$

$$x'(0) = 4.$$

1d) Find the general solution to the first order homogeneous DE in this problem, using the eigendata method we talked about last week. Note the following correspondences, which verify the discussion in the previous parts:

(i) The first component $x_1(t)$ is the general solution of the original second order homogeneous DE that we started with.

(ii) The eigenvalue "characteristic equation" for the first order system is the same as the "characteristic equation" for the second order DE.

(iii) The "Wronskian matrix" for the first order system is a "Wronskian matrix" for the second order DE.

$$(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$

$$= \lambda(7+\lambda) + 6$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1) \leftarrow \text{"same" charac poly as for DE}$$

$$\text{evals } \lambda = -1, -6.$$

$$E_{\lambda=-1} = \text{Nul} \begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad 1 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \vec{0}$$

$$E_{\lambda=-6} = \text{Nul} \begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -6 \end{bmatrix} \right\} \quad 1 \cdot \begin{bmatrix} 6 \\ -6 \end{bmatrix} - 6 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$$

$$\vec{x}_H(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-6t} \\ -c_1 e^{-t} - 6c_2 e^{-6t} \end{bmatrix}$$

for IVP in (b),

$$\text{Soln was, from above} \begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix} = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

same as W for 2nd order DE

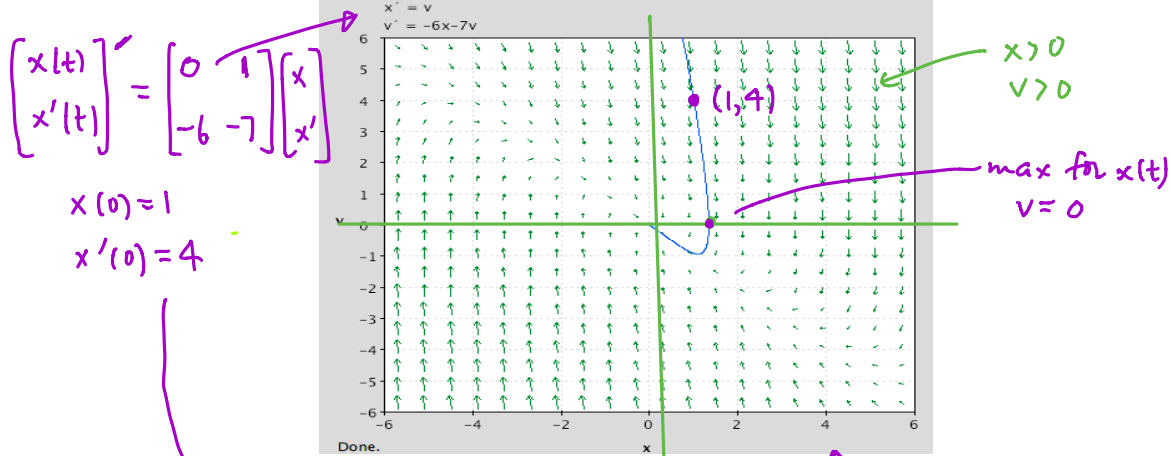
$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \rightarrow$$

$$= 2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} - 1 \begin{bmatrix} e^{-6t} \\ -6e^{-6t} \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-6t} \\ -e^{-t} & -6e^{-6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

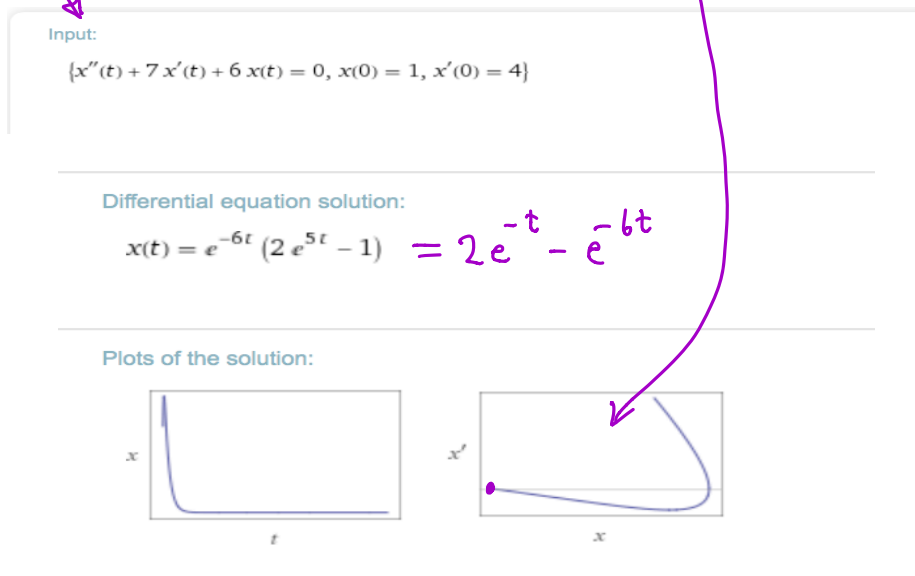
discussed on Tuesday 3/5

Pictures of the phase portrait and solution curve for the system in 1b, which is tracking position and velocity of the solution to 1a. It's possible to understand the geometry of solution curves in terms of the eigendata of the coefficient matrix, as we'll demonstrate.

From pplane, for the system:

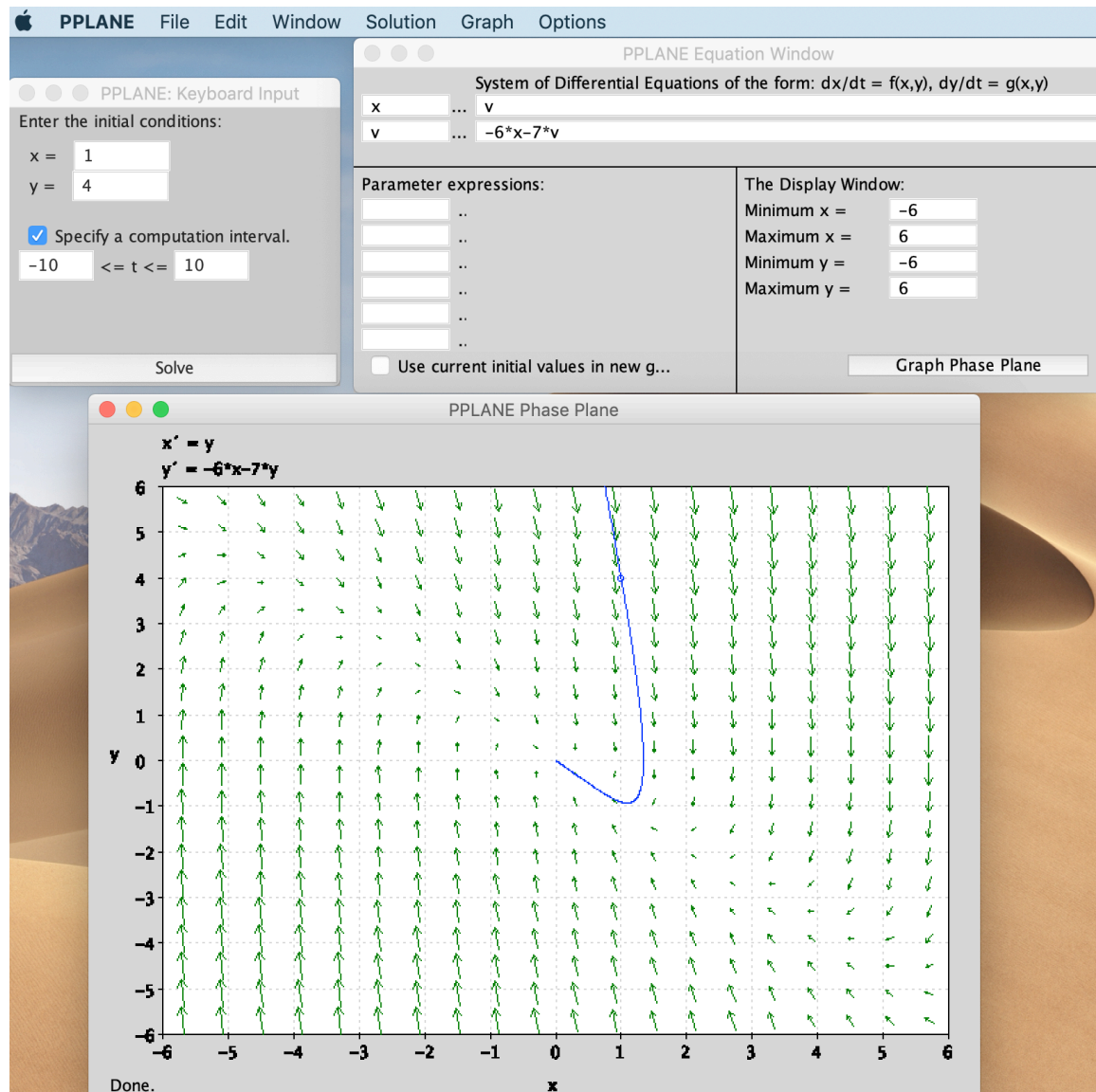


From Wolfram alpha, for the ~~underdamped~~ ^{overdamped} second order DE in 3a.



We'll demo "pplane". If you don't want to download it to your laptop (from the same URL that had "dfield"), you can just type "pplane" into a terminal window on the Math Department computer lab computers, and a cloned version of pplane will open. Ask lab assistants (or me) for help, if necessary.

Here's one screen shot for the system we've focused on, with several of the windows open. There are other interesting visualization options available in pplane that should be helpful for understanding what's going on, as I'll demonstrate in class.



General case of converting a single n^{th} order differential equation for a function $x(t)$ into a first order system of differential equations:

Write the n^{th} order DE in the form:

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t))$$

and the initial conditions as

$$\begin{aligned} x(t_0) &= b_0 \\ x'(t_0) &= b_1 \\ x''(t_0) &= b_2 \\ &\vdots \\ x^{(n-1)}(t_0) &= b_{n-1}. \end{aligned}$$

Exercise 2a) Show that if $x(t)$ solves the IVP above, then the vector function

$[x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)]^T$ solves the first order system of DE's IVP

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \dots, x_{n-1}) \\ x_1(t_0) &= b_0 \\ x_2(t_0) &= b_1 \\ x_3(t_0) &= b_2 \\ &\vdots \\ x_n(t_0) &= b_{n-1} \end{aligned}$$

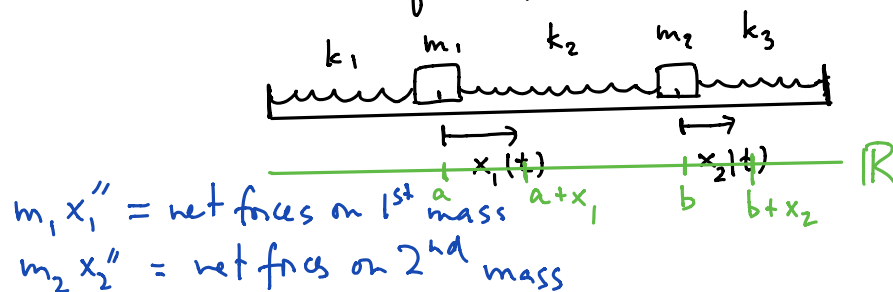
2b) (reversibility): Show that if $[x_1(t), x_2(t), \dots, x_n(t)]$ is a solution to the IVP in 2a, then the first function $x_1(t)$ solves the original IVP for the n^{th} order differential equation.

Tues Mar 5

5.2 Linear systems of DE's with complex eigendata

Announcements: w8.4e } Monday notes
pplane }
then maybe begin Tues. notes

Warm-up Exercise: This is HW 4.1.30. We'll find the DE for $x_1(t)$, which is half of the problem.



$[k_1, k_2, k_3 \text{ are linearization constants (Hookes)}$

$x_1(t), x_2(t)$ are the displacements of m_1, m_2 from their "equilibrium" positions, i.e. when the system is at rest.

e.g. $m_1 x_1'' = 0 + F_{\text{spring 1}} + F_{\text{spring 2}}$

\uparrow net forces at equil. \uparrow extra force as x_1 change \uparrow extra forces

$m_1 x_1'' = -k_1 x_1 + k_2 (\text{spring 2 stretch})$
 $= -k_1 x_1 + k_2 (x_2 - x_1)$

\vdots

\uparrow note if $x_2 = x_1$, middle has not extra stretching makes sense

use \star line to measure the new dist b/w masses

$= (b + x_2) - (a + x_1)$
 $= (b - a) + (x_2 - x_1)$

stretch at equilibrium \quad change in stretch of 2nd spring

- We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\mathbf{x}' = A \mathbf{x}.$$

So far we've not considered the possibility of complex eigenvalues and eigenvectors. As you probably touched on in Math 2270, linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc, using complex number weights. Then the model space is \mathbb{C}^n rather than \mathbb{R}^n .

Definition: $\mathbf{v} \in \mathbb{C}^n$ ($\mathbf{v} \neq \mathbf{0}$) is a complex eigenvector of the matrix A , with eigenvalue $\lambda \in \mathbb{C}$ if $A \mathbf{v} = \lambda \mathbf{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work some examples. We'll do one example relevant to the postponed homework problem, and then an interesting application to math biology. There's a general discussion at the end, for reference.

For example, consider this second order underdamped IVP for $x(t)$:

$$\begin{cases} x'' + 2x' + 5x = 0 \\ x(0) = 4 \\ x'(0) = -4 \end{cases}$$

Exercise 1)

1a) Suppose that $x(t)$ solves the IVP above. Show that $[x(t), x'(t)]^T$ solves the first order system initial value problem below as in discussions yesterday.

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \end{cases}$$

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix}$$

$$x'' = -5x - 2x'$$

$$\Rightarrow \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}' = \begin{bmatrix} x' \\ -5x - 2x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\& \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

1b) Solve the second order IVP in order to deduce the solution to the first order IVP in 5a.

$$x(t) = 4e^{-t} \cos 2t \quad \text{solves 2nd order DE.}$$

$$\text{sol is } \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 4e^{-t} \cos 2t \\ 4(-e^{-t}) \cos 2t - 8e^{-t} \sin 2t \end{bmatrix}$$

$$= 4e^{-t} \begin{bmatrix} \cos 2t \\ -\cos 2t - 2 \sin 2t \end{bmatrix}$$

1c) Start trying to solve the first order system directly, using eigenvalues and eigenvectors. Get at least as far as the roots of the characteristic polynomial. You'll finish this problem in your homework in order to practice working with complex eigenvectors, Euler's formula, and with the fact that higher order DE's correspond to certain first order systems of DE's. (It's postponed until after break.) You'll eventually recover the solution 1b as the first entry in the vector function solution to 1a.

In HW : Solve $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

find eigenvals of A

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 5 = \lambda^2 + 2\lambda + 5$$

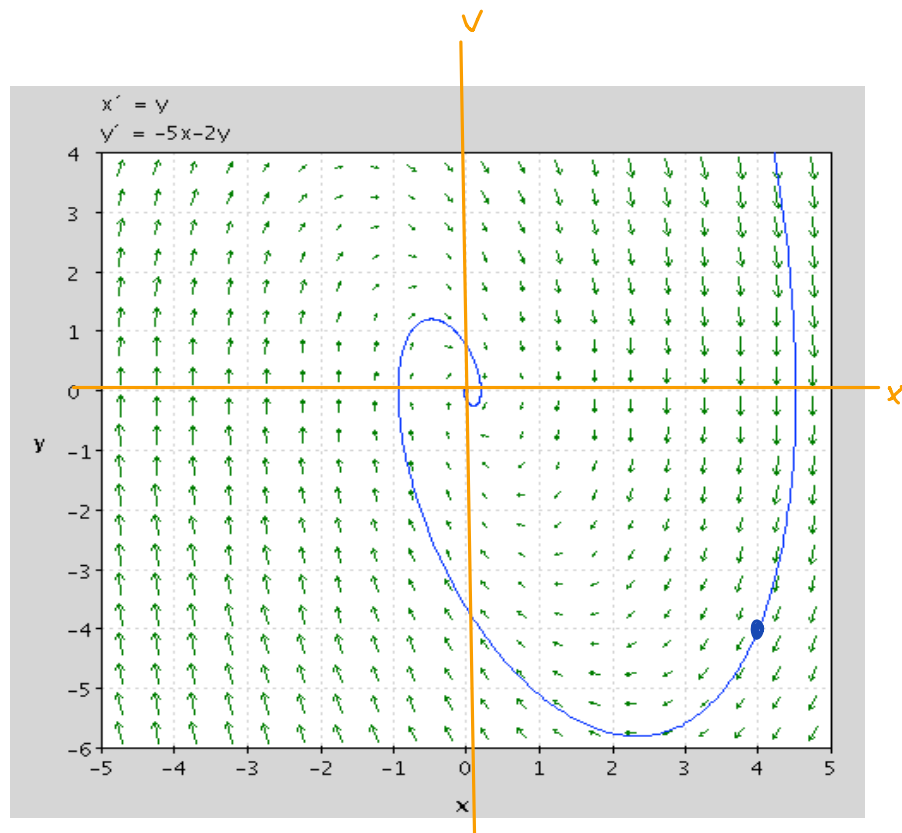
$$\text{roots } \lambda = -1 \pm 2i$$

using eigendata method.

then $x_1(t)$ will solve 2nd order DE

$$(p(r) = r^2 + 2r + 5) \text{ for DE!}$$

1d) This phase portrait and solution curve is for $[x(t), x'(t)]^T = [x(t), v(t)]^T$ from the original second order DE, corresponding to the first order system we were discussing. Interpret the solution curve in terms of the underdamped motion.



Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} -.1G -.4H \\ .1G -.1H \end{bmatrix}$$

Exercise 1a) Understand why the signs of the matrix entries are reasonable.

$a_{11} = -0.1 < 0$: if $H = 0$ & if $G > 0$, $G'(t) < 0$ because background insulin metabolizes excess glucose
 $a_{12} = -0.4 < 0$ if $H > 0$ then glucose metabolized more quickly
 $a_{21} = 0.1 > 0$ if excess glucose, pancreas notices, secretes more insulin

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$a_{22} = -0.1 < 0$, background glucose is causing insulin to get used.

1b) The first step is to get the eigendata of the matrix. Do this, and compare with the Wolfram alpha check on the next page.

$$\begin{vmatrix} -.1 - \lambda & -.4 \\ .1 & -.1 - \lambda \end{vmatrix} = (\lambda + .1)^2 + .04 = 0$$

$$(\lambda + .1)^2 = -.04$$

$$\lambda + .1 = \pm .2i$$

$$\lambda = -.1 \pm .2i$$

$$\lambda = -.1 + .2i$$

$$\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \\ \hline 10R_2 & 1 & -2i & 0 \\ 5R_1 & -i & -2 & 0 \\ \hline & 1 & -2i & 0 \\ iR_1 + R_2 & 0 & 0 & 0 \\ \rightarrow R_2 & & & \end{array}$$

$$i(-2i) - 2 = 2 - 2 = 0!$$

So,

$$E_{\lambda = -.1 + .2i} = \text{span} \left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} \right\}$$

NOTE:

For the conjugate eigenvalue $\lambda = -.1 - .2i$ we can use the conjugate eigenvector $\begin{bmatrix} -2i \\ 1 \end{bmatrix}$.

$$\lambda = -.1 - .2i$$

$$\begin{array}{cc|c} .2i & -.4 & 0 \\ .1 & .2i & 0 \\ \hline 10R_2 \rightarrow R_1 & 1 & 2i & 0 \\ 5R_1 \rightarrow R_2 & i & -2 & 0 \\ \hline & 1 & 2i & 0 \\ -iR_1 + R_2 & 0 & 0 & 0 \\ \rightarrow R_2 & & & \end{array}$$

$$\text{So } E_{\lambda = -.1 - .2i} = \text{span} \left\{ \begin{bmatrix} -2i \\ 1 \end{bmatrix} \right\}$$

Because, It is always true that if A is a matrix with real entries and

$$A\vec{v} = \lambda\vec{v}$$

then $A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

eigenvalues $\{-1, -4\}, \{1, -1\}$

Input:

eigenvalues $\begin{pmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{pmatrix}$

Results:

$\lambda_1 = -\frac{1}{10} + \frac{i}{5}$

$\lambda_2 = -\frac{1}{10} - \frac{i}{5}$

Corresponding eigenvectors:

$\vec{v}_1 = (2i, 1)$

$\vec{v}_2 = (-2i, 1)$

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$$e^{\lambda t} \vec{v}$$

so get complex soltns

$$e^{(-.1 + .2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix}, e^{(-.1 - .2i)t} \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} G' \\ H' \end{bmatrix} = \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\vec{X}'(t) = e^{-.1t} (\cos(.2t) + i \sin(.2t)) \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-.1t} (\cos(.2t) + i \sin(.2t)) 2i \\ e^{-.1t} (\cos(.2t) + i \sin(.2t)) \end{bmatrix}$$

$$= e^{-.1t} \begin{bmatrix} -2 \sin(.2t) \\ \cos(.2t) \end{bmatrix} + i e^{-.1t} \begin{bmatrix} 2 \cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$e^{\lambda t} \vec{v} = \vec{X}(t) = \vec{u}(t) + i \vec{v}(t)$$

$$\vec{X}'(t) = A \vec{X} \\ (= \lambda e^{\lambda t} \vec{v})$$

$$\vec{X}'(t) = \vec{u}'(t) + i \vec{v}'(t)$$

$$A \vec{X} = A \vec{u} + i A \vec{v}$$

so real & imag parts of \vec{X}' & $A \vec{X}$ must match

as with real eigendata

1d) Solve the initial value problem.

(using conjugate data
gives equivalent results)

So circled real vector fns

$\{\vec{u}(t), \vec{v}(t)\}$ are a basis

$$\begin{bmatrix} G(t) \\ H(t) \end{bmatrix} = c_1 e^{-.1t} \begin{bmatrix} -2 \sin(.2t) \\ \cos(.2t) \end{bmatrix} + c_2 e^{-.1t} \begin{bmatrix} 2 \cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$\text{@ } t=0 \quad \begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{array}{l} 100 = 2c_2 \\ 0 = c_1 \end{array} \quad \begin{array}{l} c_2 = 50 \\ c_1 = 0 \end{array}$$

$$\begin{bmatrix} G(t) \\ H(t) \end{bmatrix} = 50 e^{-.1t} \begin{bmatrix} 2 \cos(.2t) \\ \sin(.2t) \end{bmatrix} = \begin{bmatrix} 100 e^{-.1t} \cos(.2t) \\ 50 e^{-.1t} \sin(.2t) \end{bmatrix}$$

turns
ellipse into
spiral "sink"

because $\| \begin{bmatrix} G \\ H \end{bmatrix} \| \rightarrow 0$.

circling an ellipse

$$\frac{G^2}{4} + H^2 = 1$$

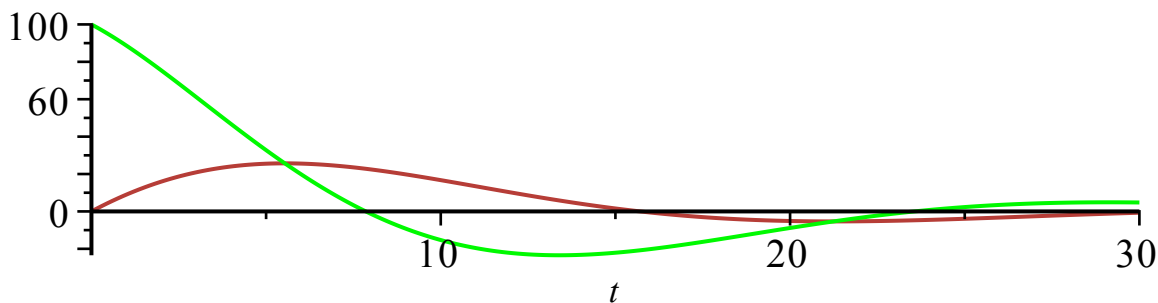
(exponentially)

Here are some pictures to help understand what the model is predicting ... using the analytic solution formulas we found.

(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t -> 100 * exp(-.1 * t) * cos(.2 * t) :
  H := t -> 50 * exp(-.1 * t) * sin(.2 * t) :
  plot1 := plot(G(t), t = 0 .. 30, color = green) :
  plot2 := plot(H(t), t = 0 .. 30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

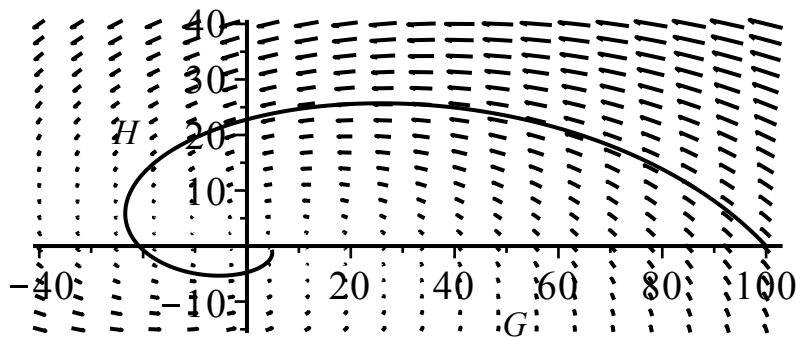
underdamped glucose-insulin interactions



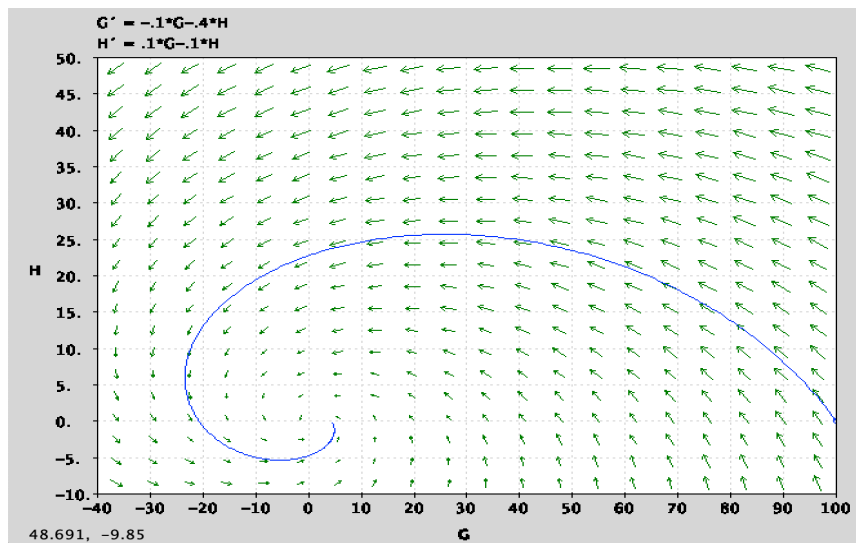
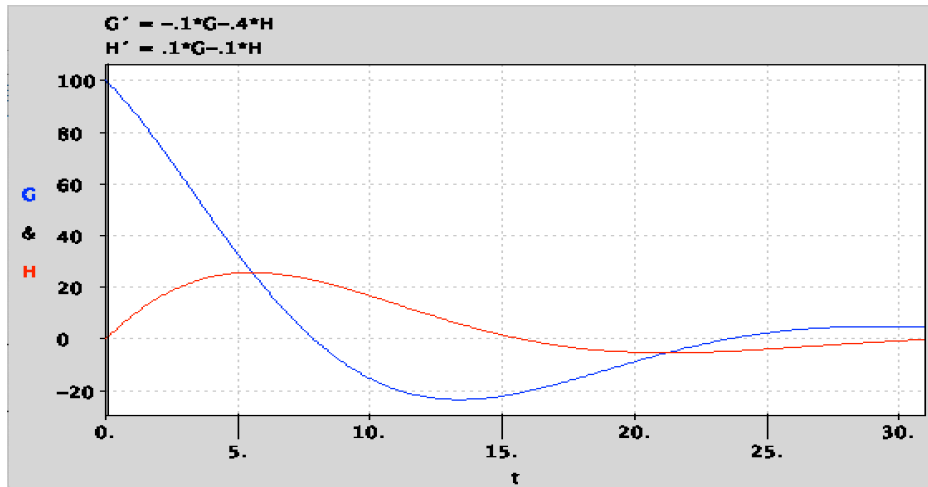
2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([- .1 * G - .4 * H, .1 * G - .1 * H], G = -40 .. 100, H = -15 .. 40) :
  soltn := plot([G(t), H(t), t = 0 .. 30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```

Glucose vs Insulin phase portrait



Same plots computed numerically using pplane. Numerical solvers for higher order differential equations IVP's convert them to equivalent first order system IVP's, and then use a Runge-Kutta type algorithm to find the solutions to the first order systems, and extract the first component function as the solution to the original differential equation IVP.



now go to Friday's

Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\mathbf{x}'(t) = A \mathbf{x}$$

Let A be a real number matrix. Let

$$\lambda = a + b i \in \mathbb{C}$$

$$\mathbf{v} = \boldsymbol{\alpha} + i \boldsymbol{\beta} \in \mathbb{C}^n$$

satisfy $A \mathbf{v} = \lambda \mathbf{v}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\mathbf{z}(t) = e^{\lambda t} \mathbf{v}$ is a complex solution to

$$\mathbf{z}'(t) = A \mathbf{z}$$

because $\mathbf{z}'(t) = \lambda e^{\lambda t} \mathbf{v}$ and this is equal to $A \mathbf{z} = A e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v}$.

- But if we write $\mathbf{z}(t)$ in terms of its real and imaginary parts,

$$\mathbf{z}(t) = \mathbf{x}(t) + i \mathbf{y}(t)$$

then the equality

$$\mathbf{z}'(t) = A \mathbf{z}$$

$$\Rightarrow \mathbf{x}'(t) + i \mathbf{y}'(t) = A(\mathbf{x}(t) + i \mathbf{y}(t)) = A \mathbf{x}(t) + i A \mathbf{y}(t).$$

Equating the real and imaginary parts on each side yields

$$\mathbf{x}'(t) = A \mathbf{x}(t)$$

$$\mathbf{y}'(t) = A \mathbf{y}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A(\boldsymbol{\alpha} + i \boldsymbol{\beta}) = (a + b i)(\boldsymbol{\alpha} + i \boldsymbol{\beta})$ then it is straightforward to check that $A(\boldsymbol{\alpha} - i \boldsymbol{\beta}) = (a - b i)(\boldsymbol{\alpha} - i \boldsymbol{\beta})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i b)t}(\boldsymbol{\alpha} - i \boldsymbol{\beta}) = \mathbf{x}(t) - i \mathbf{y}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real solutions look like:

$$\lambda = a + b i \in \mathbb{C}$$

$$\mathbf{v} = \boldsymbol{\alpha} + i \boldsymbol{\beta} \in \mathbb{C}^n$$

$$\Rightarrow e^{\lambda t} \mathbf{v} = e^{a t} (\cos(b t) + i \sin(b t)) \cdot (\boldsymbol{\alpha} + i \boldsymbol{\beta}) = \mathbf{x}(t) + i \mathbf{y}(t).$$

So the real-valued vector-valued solution functions that we'll use are

$$\mathbf{x}(t) = e^{a t} (\cos(b t) \boldsymbol{\alpha} - \sin(b t) \boldsymbol{\beta})$$

$$\mathbf{y}(t) = e^{a t} (\cos(b t) \boldsymbol{\beta} + \sin(b t) \boldsymbol{\alpha})$$

Complex conjugation for scalars and consequences for matrix/vector products.

March 8, 2019

$$\operatorname{Re} z = a, \operatorname{Im} z = b$$

For $z = a + b i$ (with a, b being real numbers) the complex conjugate \bar{z} is defined by $\bar{z} = a - b i$.

This algebraic operation is useful in a number of contexts, because it behaves nicely with respect to sums and products of complex numbers:

For $z = a + b i, w = c + d i$ with $a, b, c, d \in \mathbb{R}$, we should check that

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} & \overline{(a + bi) + (c + di)} &= \overline{(a+c) + i(b+d)} \\ & & &= (a+c) - i(b+d) \\ & & &= (a - ib) + (c - id) \\ & & &= \bar{z} + \bar{w}. \\ \overline{zw} &= \overline{(a+bi)(c+di)} = \overline{(ac-bd) + i(ad+bc)} \\ &= (ac-bd) - i(ad+bc) \\ \bar{z} \bar{w} &= (a-bi)(c-di) = ac-bd + i(-bc-cd) \end{aligned}$$

You compute the complex conjugate of a matrix or vector by computing the conjugate of each entry. Since the individual entries of the sum of two matrices are just the sums of the corresponding entries, it follows that if A, B are two matrices of the same size with complex entries, then

$$\overline{A + B} = \bar{A} + \bar{B}. \quad \checkmark$$

When you compute the product of two matrices/vectors (or of a scalar times a matrix/vector) entry by entry, you add a bunch of products where each product consists of one term from the first factor, and one term from the second. Using the fact that conjugates of sums are sums of conjugates, and conjugates of products are products of conjugates, it follows that

$$\begin{aligned} \overline{AB} &= \bar{A} \bar{B}. & \text{using above} \\ \overline{\begin{bmatrix} 2i+1 & 0 \\ 3-i & 6 \end{bmatrix} \begin{bmatrix} i \\ 2+i \end{bmatrix}} &= \overline{\begin{bmatrix} (2i+1)(i) \\ (3-i)(i) + 6(2+i) \end{bmatrix}} = \begin{bmatrix} (-2i+1)(-i) \\ (3+i)(-i) + 6(2-i) \end{bmatrix} \\ \bar{A} \bar{B} &= \begin{bmatrix} -2i+1 & 0 \\ 3+i & 6 \end{bmatrix} \begin{bmatrix} -i \\ 2-i \end{bmatrix} \end{aligned}$$

Special case: If A is a matrix with real number entries and v is a complex eigenvector with complex eigenvalue λ , i.e

$$A \vec{v} = \lambda \vec{v},$$

then

$$\overline{A \vec{v}} = \overline{\lambda \vec{v}}$$

$$A \vec{\bar{v}} = \bar{\lambda} \vec{\bar{v}}.$$

This is why the conjugate of the eigenvector is an eigenvector for the conjugate eigenvalue!

no back to G-H model.

5.3 Begin phase diagrams for two linear systems of first order differential equations

Announcements: • today: complex eigen data in A : $\vec{x}'(t) = A\vec{x}$.
 & 2nd order DE's \rightarrow 1st order systems or 2 DE's

Tuesday's notes

Warm-up Exercise: Do enough checking to verify that the solution to

$$\begin{cases} x'' + 2x' + 5x = 0 \\ x(0) = 4 \\ x'(0) = -4 \end{cases}$$

is $x(t) = 4e^{-t}\cos 2t$... it'll save time in class if we know this

check: $p(r) = r^2 + 2r + 5 = 0$

$$(r+1)^2 + 4 = 0$$

$$(r+1)^2 = -4$$

$$r+1 = \pm 2i$$

$$r = -1 \pm 2i$$

$$\Rightarrow x_H(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t \quad \checkmark$$

if $x(t) = 4e^{-t}\cos 2t$

$$\Rightarrow x(0) = 4 \quad \checkmark$$

$$x'(0) = -4 \quad \checkmark$$

5.3 phase diagrams for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A . This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points.

We will consider the cases of real eigenvalues and complex eigenvalues separately.

Real eigenvalues If the matrix $A_{2 \times 2}$ is diagonalizable, i.e. if there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of \mathbb{R}^2 consisting of eigenvectors of A , then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

- And, for each $\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$ the value of the tangent field at \mathbf{x} is

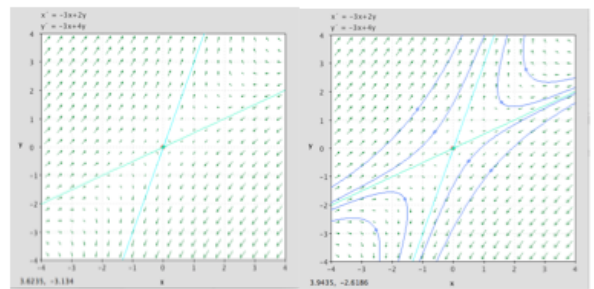
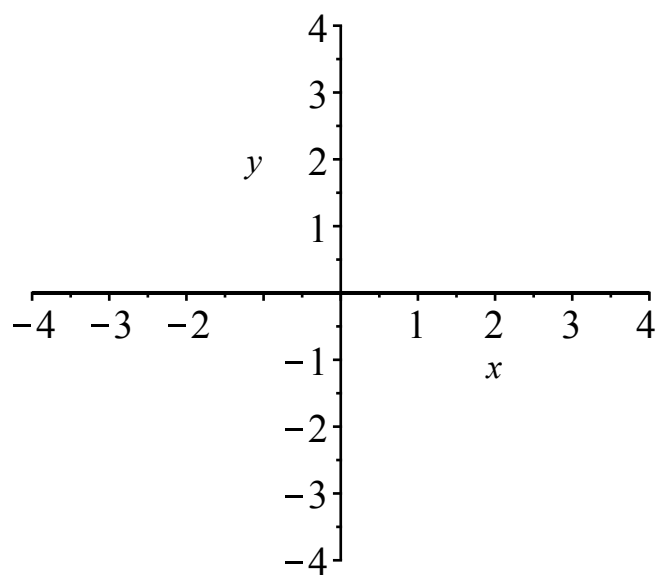
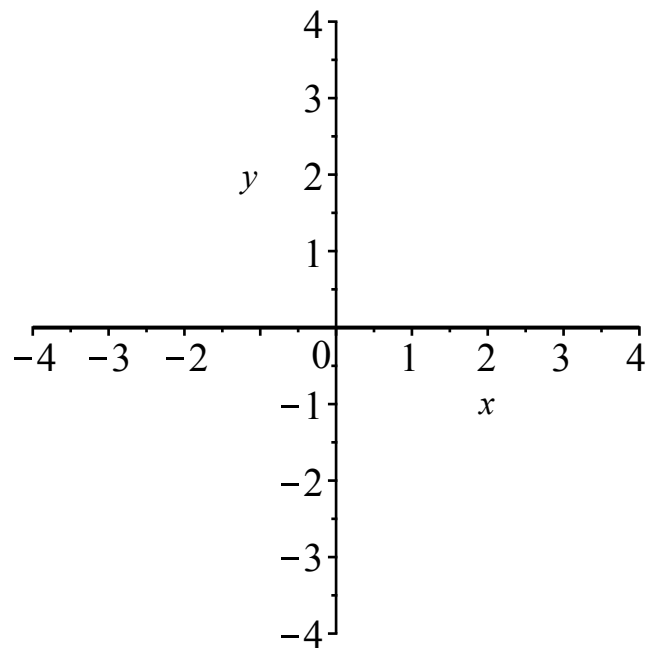
$$A\mathbf{x} = A(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2.$$

(The text discusses the case of non-diagonalizable A . This can only happen if $\det(A - \lambda I) = (\lambda - \lambda_1)^2$, but the dimension of the $\lambda = \lambda_1$ eigenspace is only one.)

Exercise 1) This is an example of what happens when A has two real eigenvalues of opposite sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

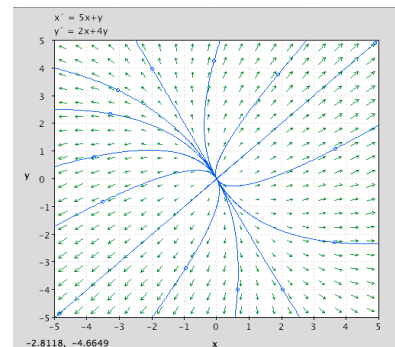
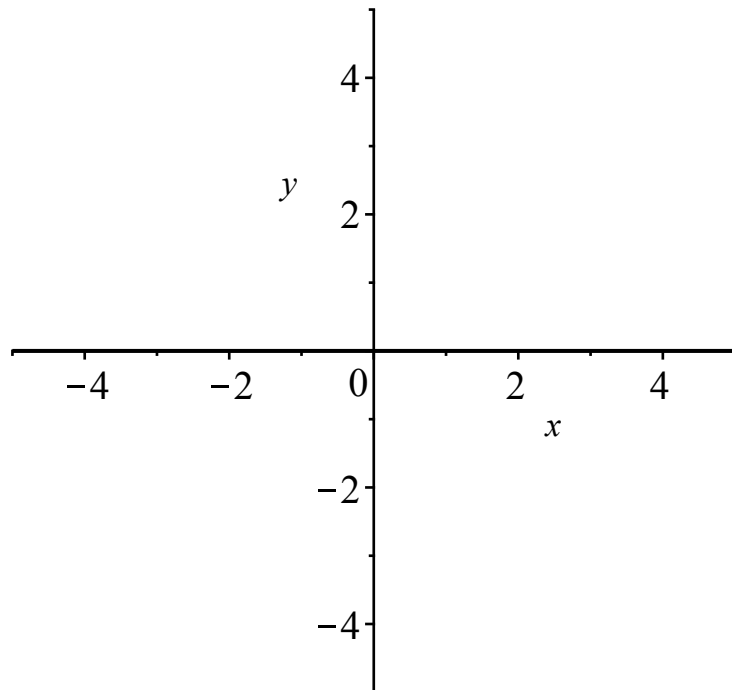
- Find the eigendata for A , and the general solution to the first order system of DE's.
 - (On the next page) use just the eigendata to sketch the tangent vector field (on the first plot). Begin by sketching the two eigenspaces.
 - (On the next page) use just the general solutions to the DE system to sketch representative solution curves (on the second plot).
- Your answers to b,c should be consistent.



Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- a) Find the eigendata for A , and the general solution to the first order system of DE's.
 b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves.



Theorem: Time reversal: If $\mathbf{x}(t)$ solves

$$\mathbf{x}' = A \mathbf{x}$$

then $\mathbf{z}(t) := \mathbf{x}(-t)$ solves

$$\mathbf{z}' = (-A)\mathbf{z}$$

proof: by the chain rule,

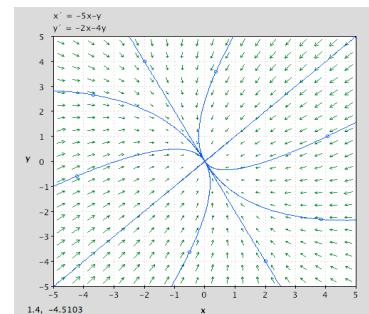
$$\mathbf{z}'(t) = \mathbf{x}'(-t) \cdot (-1) = -\mathbf{x}'(-t) = -A \mathbf{x}(-t) = -A \mathbf{z}.$$

Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A ?

b) Describe how the eigendata of the matrix in the system below, is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



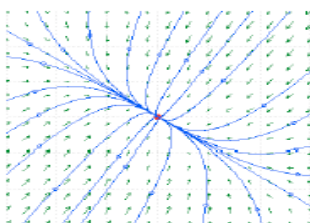
summary: In case the matrix $A_{2 \times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\mathbf{x}'(t) = A \mathbf{x}$$

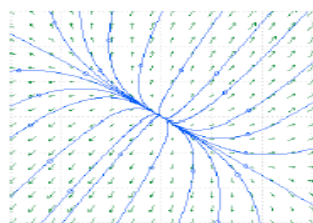
has general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

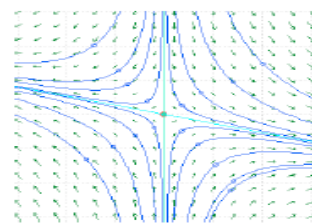
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
 $\lambda_1, \lambda_2 < 0$



nodal source
 $\lambda_1, \lambda_2 > 0$



saddle point
 $\lambda_1 < 0 < \lambda_2$

Math 2280-002

Fri Mar 8

5.3, phase portraits for complex eigendata; Introduction to 6.1: phase diagrams for two linear systems of *nonlinear* first order differential equations

Announcements:



today.

(First finish glucose insulin model, then
(skip Wed notes, do Friday's, complete Wed
after break)

- also, handout on complex conjugation algebra for scalars & for matrix/vector sums & products

Warm-up Exercise:

not today. 😊

complex eigenvalues: Consider the first order system

$$\mathbf{x}'(t) = A \mathbf{x}$$

Let $A_{2 \times 2}$ have complex eigenvalues $\lambda = p \pm q i$. For $\lambda = p + q i$ let the eigenvector be $\mathbf{y} = \mathbf{a} + \mathbf{b} i$.

Then we know that we can use the complex solution $e^{\lambda t} \mathbf{y}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{\lambda t} \mathbf{y} = e^{(p+qi)t} (\mathbf{a} + \mathbf{b} i) \\ &= e^{p t} (\cos(q t) + i \sin(q t)) (\mathbf{a} + \mathbf{b} i) \\ &= [e^{p t} \cos(q t) \mathbf{a} - e^{p t} \sin(q t) \mathbf{b}] \\ &\quad + i [e^{p t} \sin(q t) \mathbf{a} + e^{p t} \cos(q t) \mathbf{b}] . \end{aligned}$$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above:

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{p t} [\cos(q t) \mathbf{a} - \sin(q t) \mathbf{b}] \\ &\quad + c_2 e^{p t} [\sin(q t) \mathbf{a} + \cos(q t) \mathbf{b}] . \end{aligned} = e^{p t} \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cos q t & \sin q t \\ -\sin q t & \cos q t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can rewrite $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = e^{p t} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

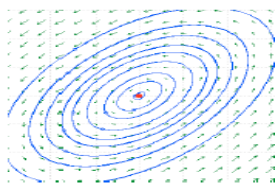
$$\begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

- transformed into a parametric ellipse by a matrix transformation of \mathbb{R}^2 :

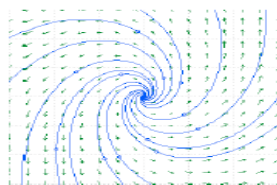
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

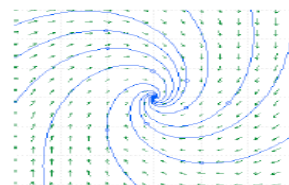
Thus $\mathbf{x}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center
 $\text{Re}(\lambda)=0$



spiral source
 $\text{Re}(\lambda)>0$



spiral sink
 $\text{Re}(\lambda)<0$

$$\begin{aligned} &\underbrace{\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cos q t & \sin q t \\ -\sin q t & \cos q t \end{bmatrix}}_{\begin{bmatrix} \cos q t \vec{a} - \sin q t \vec{b} & \sin q t \vec{a} + \cos q t \vec{b} \end{bmatrix}} \\ &\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &\quad (2270) \end{aligned}$$

Exercise 1) Do the eigenvalue analysis, ~~find the general solution~~, and use tangent vectors just along the two axes to sketch typical solution curve trajectories, for this system from your postponed homework problem.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 5$$

$$= \lambda^2 + 2\lambda + 5$$

$$= (\lambda+1)^2 + 4 = 0$$

$$(\lambda+1)^2 = -4$$

$$\lambda+1 = \pm 2i$$

$$\lambda = -1 \pm 2i$$

$$e^{\lambda t} \vec{v}$$

$$e^{(-1+2i)t} \vec{v}$$

$$e^{(-1-2i)t} \vec{\bar{v}}$$

e^{-t} factor says spiral sink.

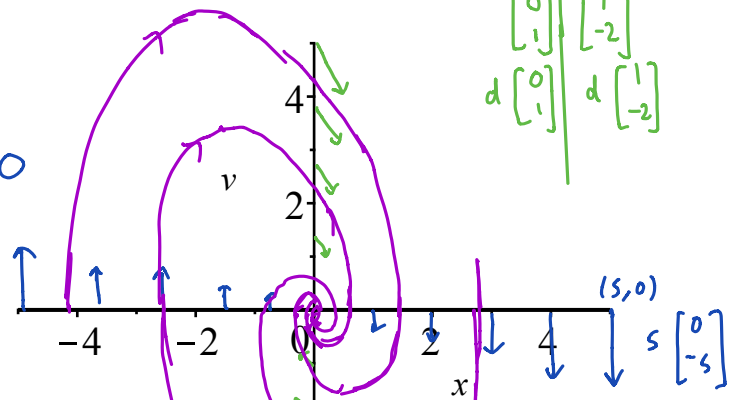
$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x' \\ v' \end{bmatrix} = A \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} c \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



because we know it's a spiral sink we can do a good job of interpolating the solution curves just with the partial tangent info along the coord axes.

