Math 2280-002
Week 9, March 4-8 5.1-5.3, introduction to Chapter 6
Mon Mar 4
5.1-5.2 Systems of differential equations - summary so far; converting differential equations (or systems) into equivalent first order systems of DEs; example with pplane visualization and demo.

Announcements:

Warm-up Exercise:

Summary of Chapters 4-5 so far (for reference):

Theorem 1   For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2   For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{f}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $P(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Theorem 3) Vector space theory for first order systems of linear DEs (We noticed the familiar themes... we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\boldsymbol{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) + P(t)\underline{x}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{z}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{z}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)) .$$

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{x}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}} .$$

3.3) (Generalizes what we talked about on Friday last week.) For $P(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the t-interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{\mathbf{0}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$ defined by

$$[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \ldots, \underline{\boldsymbol{X}}_n)](t_0) := [\underline{\boldsymbol{X}}_1(t_0)|\underline{\boldsymbol{X}}_2(t_0)|\cdots|\underline{\boldsymbol{X}}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible Wronskian matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{\mathbf{0}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + \ldots + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients comprise the solution vector to the Wronskian matrix equation

$$\begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, the vector functions $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\mathbf{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\mathbf{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and the number n is the dimension of the solution space.

New today: Along with the sum rule and constant scalar multiple rule for vector (or matrix) valued functions, we will be using the Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\boldsymbol{x}}(t)$ (scalar function times vector function),

$f(t)A(t)$ (scalar function times matrix function),

$A(t)\underline{\boldsymbol{x}}(t)$ (matrix function times vector function),

$\underline{\boldsymbol{x}}(t) \bullet \underline{\boldsymbol{y}}(t)$ (vector function dot product with vector function),

$\underline{\boldsymbol{x}}(t) \times \underline{\boldsymbol{y}}(t)$ (cross product of two vector functions),

$A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t), B(t)$ be differentiable scalar, matrix or vector-valued functions of t, and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t)*B(t)) = A'(t)*B(t) + A(t)*B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. There is one explanation that proves all of those product rules at once:

Theorem 4) The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and the tank example): For the homogeneous system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

with $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$, $A_{n\times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ of $\mathbb{R}^n$ made out of eigenvectors of A, i.e. $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ for each $j = 1, 2, \ldots, n$), then the functions

$$e^{\lambda_j t}\,\underline{\boldsymbol{v}}_j\,, \quad j = 1, 2, \ldots n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\underline{\boldsymbol{x}}_H(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \ldots + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n\,.$$

proof: check the Wronskian matrix at $t = 0$, it's the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

There is an alternate direct proof of Theorem 4 which is based on diagonalization from Math 2270. You are asked to use the idea of this alternate proof to solve a nonhomogeneous linear system of DE's this week - in homework problem w8.4e.

proof 2:

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the invertible matrix with those eigenvectors as columns:

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$
$$A = P\,\mathrm{D}\,P^{-1}$$
$$P^{-1}A\,P = \mathrm{D}\,.$$

Now let's feed this into our system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

let's change functions in our DE system:

$$\underline{\boldsymbol{x}}(t) = P\,\underline{\boldsymbol{u}}(t), \qquad (\ \underline{\boldsymbol{u}}(t) = P^{-1}\underline{\boldsymbol{x}}(t)\ )$$

and work out what happens (and see what would happen if the system of DE's was non-homogeneous, as in your homework).

New today: It is always the case that an initial value problem for one or more differential equations of arbitrary order is equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example. (See examples and homework problems in section 4.1) This gives us a new perspective on the way we solved differential equations from Chapter 3.

For example, consider this overdamped problem from Chapter 3:

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

Exercise 1a) Do enough checking to verify that $x(t) = 2e^{-t} - e^{-6t}$ is the solution to this IVP.

1b) Show without using the formula for its solution, that if $x(t)$ solves the IVP above, then the vector function $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Then use the solution from 1a to write down the solution to the IVP in 1b.

1c) Show that if $\left[x_1(t), x_2(t)\right]^T$ solves the IVP in 1b then without using the formula for the solution, show that the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure (when the system arises that way).

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x''(t) + 7\,x'(t) + 6\,x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

1d) Find the general solution to the first order homogeneous DE in this problem, using the eigendata method we talked about last week. Note the following correspondences, which verify the discussion in the previous parts:

(i) The first component $x_1(t)$ *is* the general solution of the original second order homogeneous DE that we started with.

(ii) The eigenvalue "characteristic equation" for the first order system *is* the same as the "characteristic equation" for the second order DE.

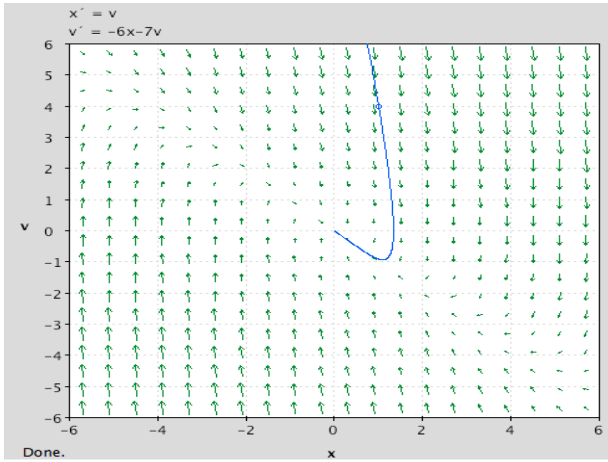
(iii) The "Wronskian matrix" for the first order system *is* a "Wronskian matrix" for the second order DE.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x''(t) + 7\,x'(t) + 6\,x(t) = 0$$

Pictures of the phase portrait and solution curve for the system in 1b, which is tracking position and velocity of the solution to 1a. It's possible to understand the geometry of solution curves in terms of the the eigendata of the coefficient matrix, as we'll demonstrate.

From pplane, for the system:



From Wolfram alpha, for the underdamped second order DE in 3a.

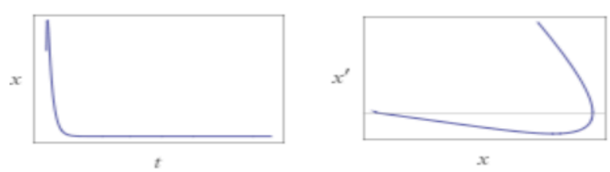
Input:

$$\{x''(t) + 7x'(t) + 6x(t) = 0,\ x(0) = 1,\ x'(0) = 4\}$$

Differential equation solution:

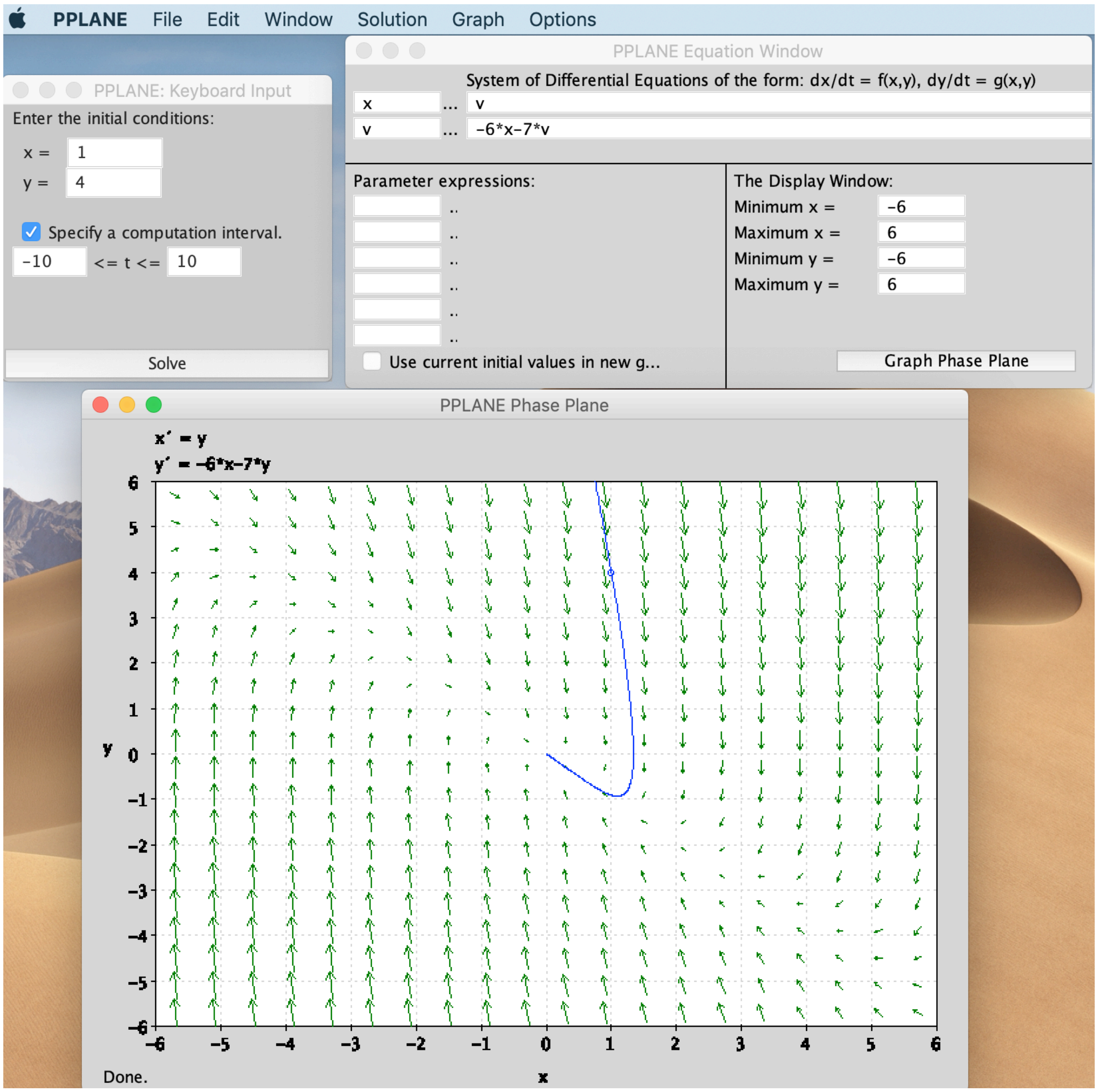
$$x(t) = e^{-6t}\,(2e^{5t} - 1)$$

Plots of the solution:



We'll demo "pplane". If you don't want to download it to your laptop (from the same URL that had "dfield"), *you can just type "pplane" into a terminal window on the Math Department computer lab computers, and a cloned version of pplane will open. Ask lab assistants (or me) for help, if necessary.*

Here's one screen shot for the system we've focused on, with several of the windows open. There are other interesting visualization options available in pplane that should be helpful for understanding what's going on, as I'll demonstrate in class.



General case of converting a single n^{th} order differential equation for a function $x(t)$ into a first order system of differential equations:

Write the n^{th} order DE in the form:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots x^{(n-1)}(t)\right)$$

and the initial conditions as

$$\begin{aligned} x(t_0) &= b_0 \\ x'(t_0) &= b_1 \\ x''(t_0) &= b_2 \\ &\vdots \\ x^{(n-1)}(t_0) &= b_{n-1}. \end{aligned}$$

Exercise 2a) Show that if $x(t)$ solves the IVP above, then the vector function $\left[x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)\right]^T$ solves the first order system of DE's IVP

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f\left(t, x_1, x_2, \ldots x_{n-1}\right) \\ x_1(t_0) &= b_0 \\ x_2(t_0) &= b_1 \\ x_3(t_0) &= b_2 \\ &\vdots \\ x_n(t_0) &= b_{n-1} \end{aligned}$$

2b) (reversibility): Show that if $\left[x_1(t), x_2(t), \ldots x_n(t)\right]$ is a solution to the IVP in 2a, then the first function $x_1(t)$ solves the original IVP for the n^{th} order differential equtation.

Tues Mar 5
5.2 Linear systems of DE's with complex eigendata

Announcements:

Warm-up Exercise:

• We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}.$$

So far we've not considered the possibility of complex eigenvalues and eigenvectors. As you probably touched on in Math 2270, linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc, using complex number weights. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{\boldsymbol{v}} \in \mathbb{C}^n$ $(\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}})$ is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work some examples. We'll do one example relevant to the postponed homework problem, and then an interesting application to math biology. There's a general discussion at the end, for reference.

For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$
$$x(0) = 4$$
$$x'(0) = -4 \ .$$

Exercise 1)

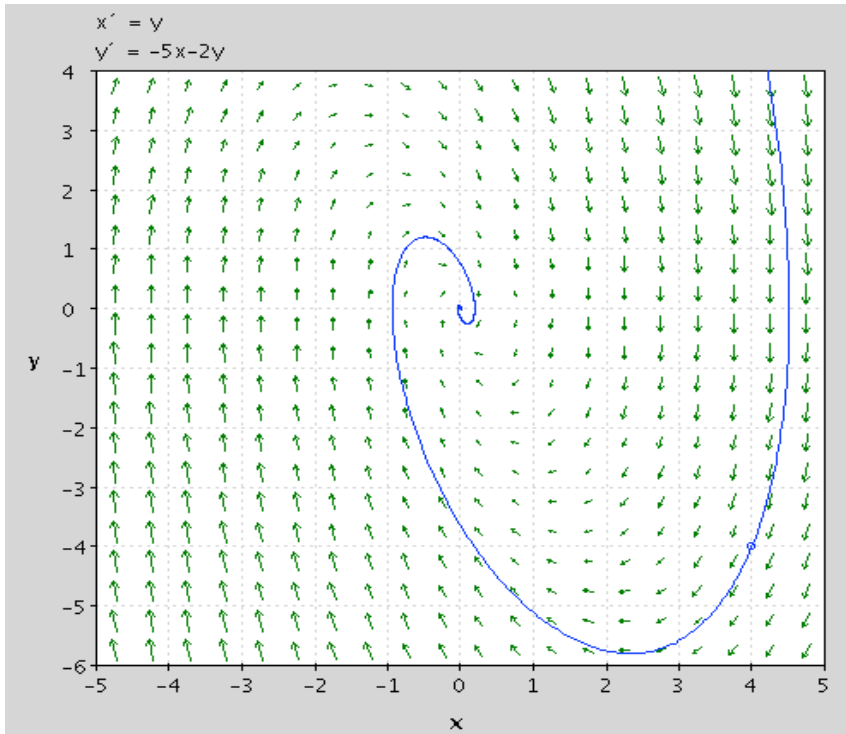
1a) Suppose that $x(t)$ solves the IVP above. Show that $[x(t), x'(t)]^T$ solves the first order system initial value problem below as in discussions yesterday.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

1b) Solve the second order IVP in order to deduce the solution to the first order IVP in 5a.

1c) Start trying to solve the first order system directly, using eigenvalues and eigenvectors. Get at least as far as the roots of the characteristic polynomial. You'll finish this problem in your homework in order to practice working with complex eigenvectors, Euler's formula, and with the fact that higher order DE's correspond to certain first order systems of DE's. (It's postponed until after break.) You'll eventually recover the solution 1b as the first entry in the vector function solution to 1a.

1d) This phase portrait and solution curve is for $[x(t), x'(t)]^T = [x(t), v(t)]^T$ from the original second order DE, corresponding to the first order system we were discussing. Interpret the solution curve in terms of the underdamped motion.



Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 1a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

1b) The first step is to get the eigendata of the matrix. Do this, and compare with the Wolfram alpha check on the next page.

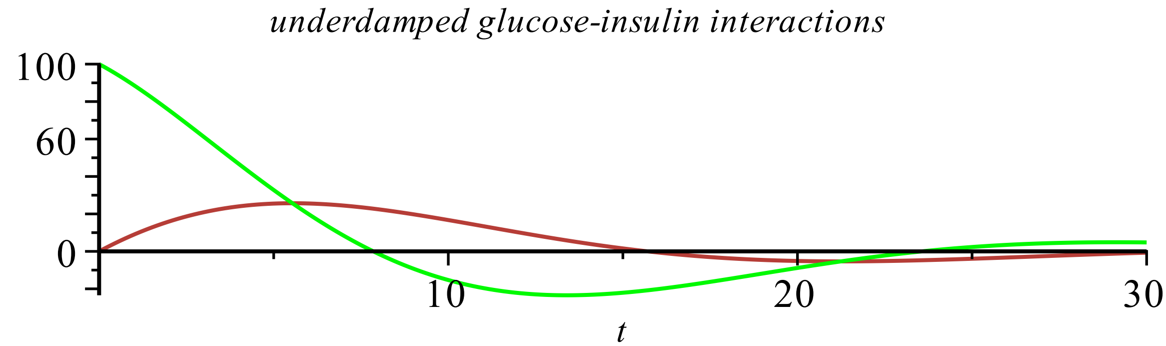
1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

1d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... using the analytic solution formulas we found.

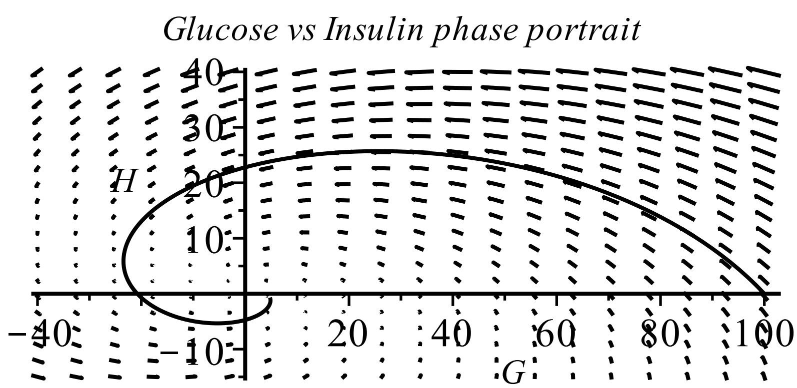
(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t→100·exp(-.1·t)·cos(.2·t) :
  H := t→50·exp(-.1·t)·sin(.2·t) :
  plot1 := plot(G(t), t = 0 ..30, color = green) :
  plot2 := plot(H(t), t = 0 ..30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

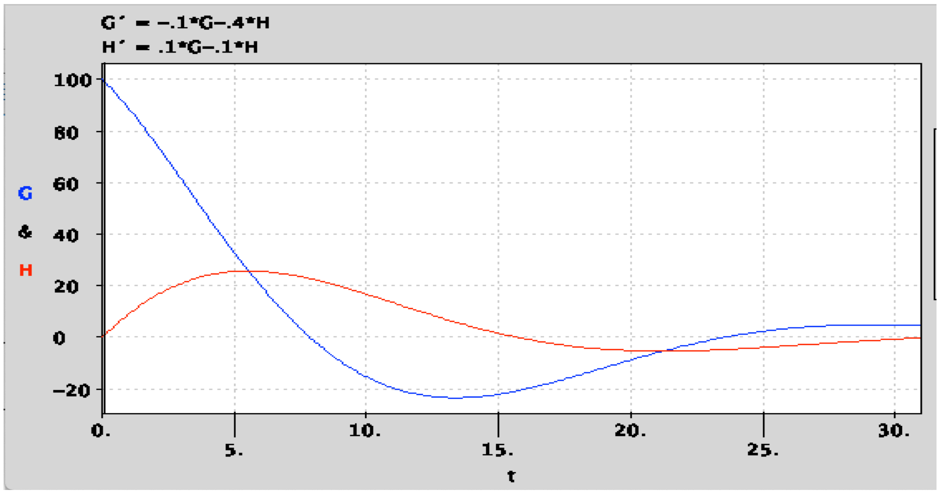


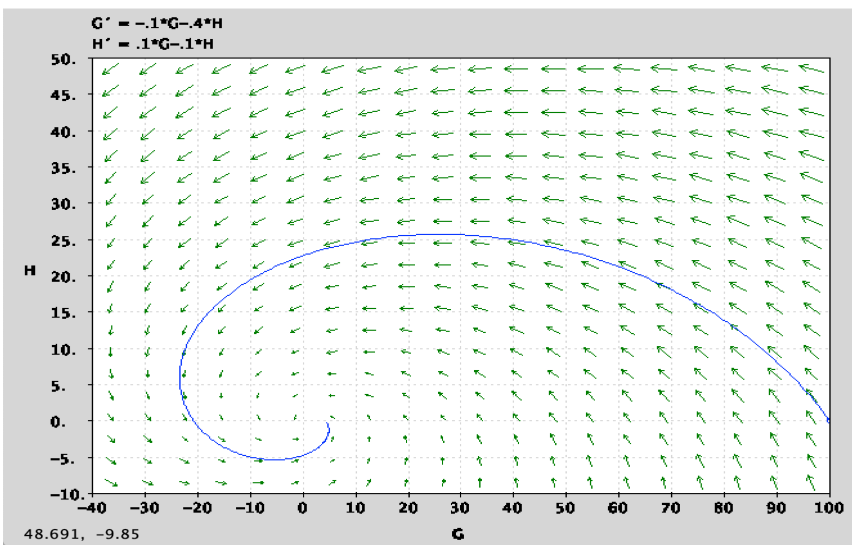
2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([-.1·G − .4·H, .1·G − .1·H], G =-40 ..100, H =-15 ..40) :
  soltn := plot([G(t), H(t), t = 0 ..30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```



Same plots computed numerically using pplane. Numerical solvers for higher order differential equations IVP's convert them to equivalent first order system IVP's, and then use a Runge-Kutta type algorithm to find the solutions to the first order systems, and extract the first component function as the solution to the original differential equation IVP.





Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\mathbf{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

satisfy $A\,\underline{\mathbf{v}} = \boldsymbol{\lambda}\,\underline{\mathbf{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t}\underline{\mathbf{v}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A\,\mathbf{x}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t}\underline{\mathbf{v}}$ and this is equal to $A\,\underline{\mathbf{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\mathbf{v}} = e^{\lambda t}A\,\underline{\mathbf{v}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)$$

then the equality

$$\underline{\mathbf{z}}'(t) = A\,\mathbf{z}$$
$$\Rightarrow \underline{\mathbf{x}}'(t) + i\,\underline{\mathbf{y}}'(t) = A(\underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t)) = A\,\underline{\mathbf{x}}(t) + i\,A\,\underline{\mathbf{y}}(t)\,.$$

Equating the real and imaginary parts on each side yields

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}(t)$$
$$\underline{\mathbf{y}}'(t) = A\,\underline{\mathbf{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\left(\underline{\alpha} + i\,\underline{\boldsymbol{\beta}}\right) = (a + b\,i)\left(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}}\right)$ then it is straightforward to check that $A\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = (a - b\,i)\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right)$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i\,b)t}\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = \underline{\mathbf{x}}(t) - i\,\underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real soluitions look like:

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\mathbf{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

$$\Rightarrow \mathrm{e}^{\lambda t}\underline{\mathbf{v}} = \mathrm{e}^{a t}(\cos(b\,t) + i\sin(b\,t))\cdot\left(\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}}\,i\right) = \underline{\mathbf{x}}(t) + i\,\underline{\mathbf{y}}(t).$$

So the real-valued vector-valued solution functions that we'll use are

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{a t}\left(\cos(bt)\underline{\boldsymbol{\alpha}} - \sin(bt)\underline{\boldsymbol{\beta}}\right)$$
$$\underline{\mathbf{y}}(t) = \mathrm{e}^{a t}\left(\cos(bt)\underline{\boldsymbol{\beta}} + \sin(bt)\underline{\alpha}\right)$$

Math 2280-002
Wed Mar 6
5.3 Begin phase diagrams for two linear systems of first order differential equations

Announcements:

Warm-up Exercise:

5.3 phase diagrams for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A. This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points.

We will consider the cases of real eigenvalues and complex eigenvalues separately.

Real eigenvalues If the matrix $A_{2\times 2}$ is diagonalizable, i.e. if there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of A), then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ is

$$x(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2$$

- And, for each $\underline{\boldsymbol{x}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2$ the value of the tangent field at x is

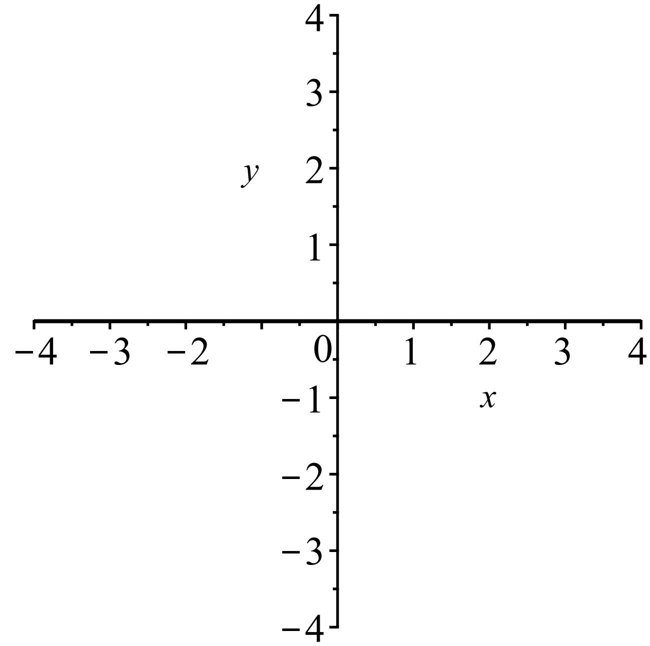
$$A\,\underline{\boldsymbol{x}} = A\left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2\right) = c_1\lambda_1\underline{\boldsymbol{v}}_1 + c_2\lambda_2\underline{\boldsymbol{v}}_2.$$

(The text discusses the case of non-diaganalizable A. This can only happen if $det(A - \lambda I) = \left(\lambda - \lambda_1\right)^2$, but the dimension of the $\lambda = \lambda_1$ eigenspace is only one.)

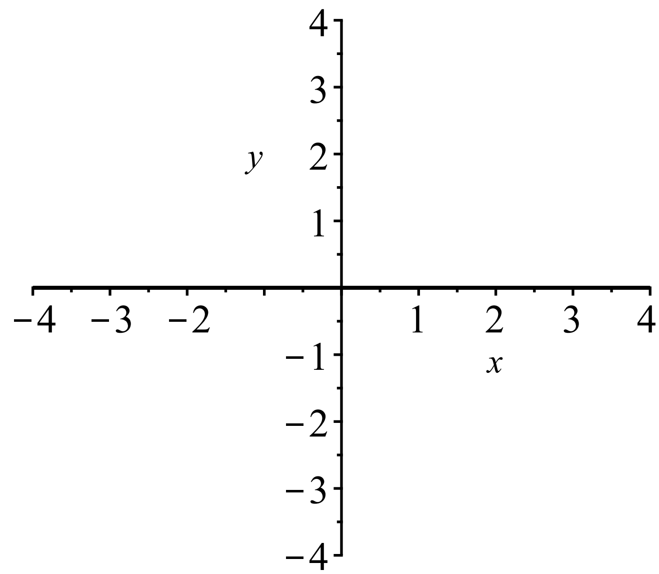
Exercise 1) This is an example of what happens when A has two real eigenvalues of opposite sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

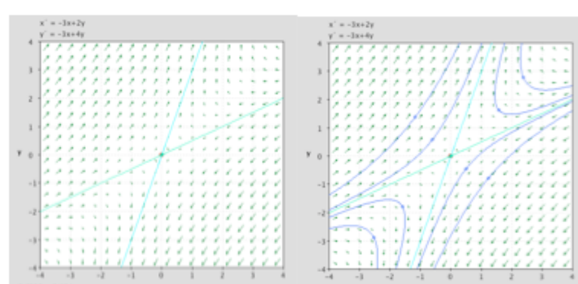
a) Find the eigendata for A, and the general solution to the first order system of DE's.
b) (On the next page) use just the eigendata to sketch the tangent vector field (on the first plot). Begin by sketching the two eigenspaces.
c) (On the next page) use just the general solutions to the DE system to sketch representative solution curves (on the second plot).
Your answers to b,c should be consistent.



y
x



y
x

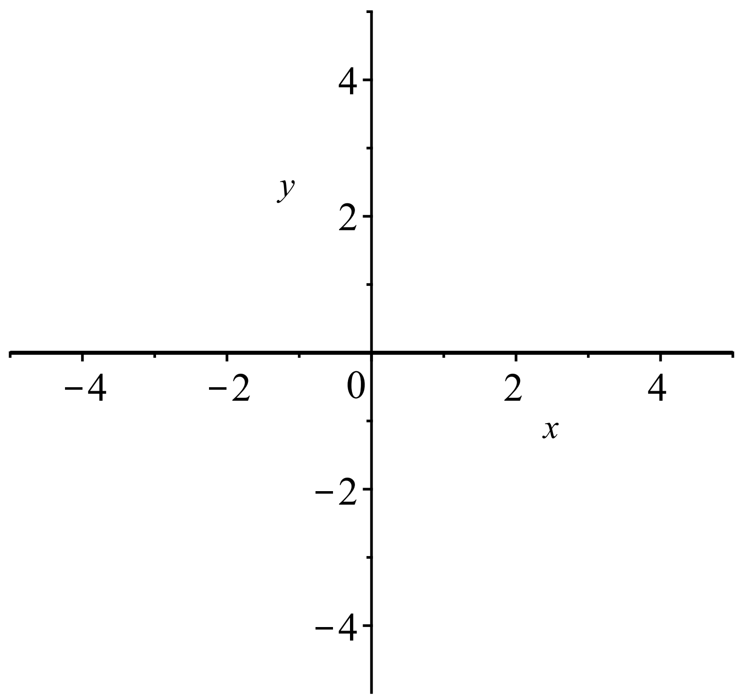


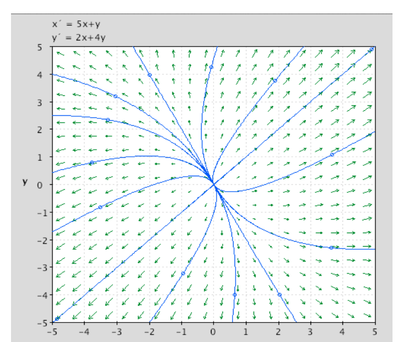
x' = -3x+2y
y' = -3x+4y
x' = -3x+2y
y' = -3x+4y

Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the eigendata for A, and the general solution to the first order system of DE's.
b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves.





Theorem: Time reversal: If $\underline{\mathbf{x}}(t)$ solves

$$\underline{\mathbf{x}}' = A\,\underline{\mathbf{x}}$$

then $\underline{\mathbf{z}}(t) := \underline{\mathbf{x}}(-t)$ solves

$$\underline{\mathbf{z}}' = (-A)\underline{\mathbf{z}}$$

proof: by the chain rule,

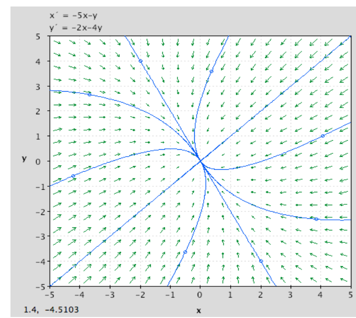
$$\underline{\mathbf{z}}'(t) = \underline{\mathbf{x}}'(-t)\cdot(-1) = -\underline{\mathbf{x}}'(-t) = -A\,\underline{\mathbf{x}}(-t) = -A\,\underline{\mathbf{z}}.$$

Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A?

b) Describe how the eigendata of the matrix in the system below, is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



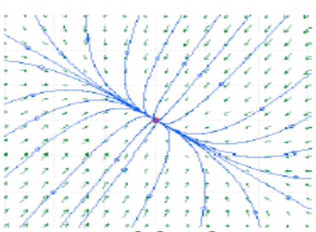
summary: In case the matrix $A_{2\times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

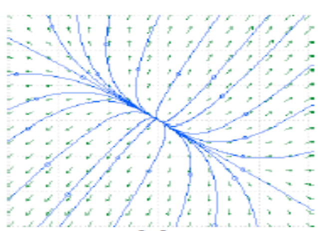
has general solution

$$\underline{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\underline{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\underline{\mathbf{v}}_2\,.$$

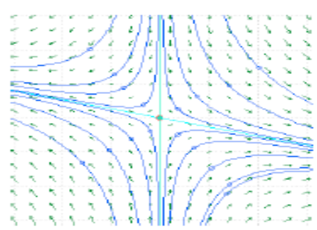
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
$\lambda_1, \lambda_2 < 0$



nodal source
$\lambda_1, \lambda_2 > 0$



saddle point
$\lambda_1 < 0 < \lambda_2$

Math 2280-002
Fri Mar 8
5.3, phase portraits for complex eigendata; Introduction to 6.1: phase diagrams for two linear systems of *nonlinear* first order differential equations

Announcements:

Warm-up Exercise:

complex eigenvalues: Consider the first order system

$$\underline{\boldsymbol{x}}'(\underline{t}) = A\,\underline{\boldsymbol{x}}$$

Let $A_{2\times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i$.

Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\begin{aligned}\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= \left[e^{p\,t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p\,t}\sin(q\,t)\underline{\boldsymbol{b}}\right]\\ &\quad + i\left[e^{p\,t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p\,t}\cos(q\,t)\underline{\boldsymbol{b}}\right].\end{aligned}$$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above:

$$\begin{aligned}\underline{\boldsymbol{x}}(t) &= c_1 e^{p\,t}[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}]\\ &\quad + c_2 e^{p\,t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}].\end{aligned}$$

We can rewrite $\underline{\boldsymbol{x}}(t)$ as

$$\underline{\boldsymbol{x}}(t) = e^{p\,t}\begin{bmatrix} a_1 & b_1\\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1\\ c_2\end{bmatrix}.$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

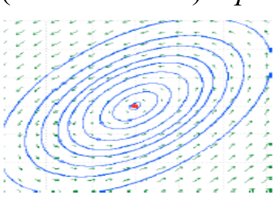
$$\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1\\ c_2\end{bmatrix}.$$

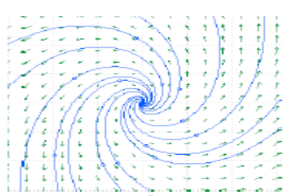
- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

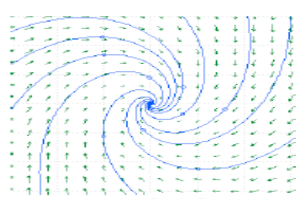
$$\begin{bmatrix} a_1 & b_1\\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1\\ c_2\end{bmatrix}.$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p\,t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

Thus $\underline{\boldsymbol{x}}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:

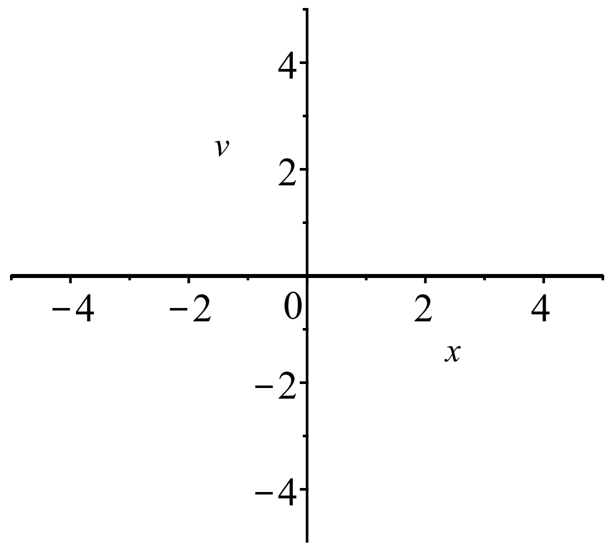

center
Re(λ)=0

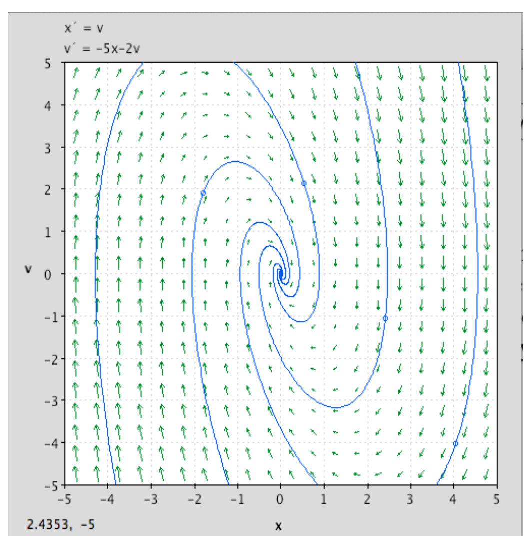

spiral source
Re(λ)>0


spiral sink
Re(λ)<0

Exercise 1) Do the eigenvalue analysis, find the general solution, and use tangent vectors just along the two axes to sketch typical solution curve trajectories, for this system from your posponed homework problem.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$





Introduction to Chapter 6. (We'll return to Chapter 5 after we discuss Chapter 6.) This chapter is about general (non-linear) system of two first order differential equations for $x(t), y(t)$ can be written as

$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$

which we often abbreviate, by writing

$$x' = F(x, y, t)$$
$$y' = G(x, y, t) \ .$$

If the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t, i.e.

$$x' = F(x, y)$$
$$y' = G(x, y)$$

then the system is called autonomous. Autonomous systems of first order DEs are the focus of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2. In Chapter 6 the text restricts to systems of two equations as above, although most of the ideas generalize to more complicated autonomous systems with three or more interacting functions.

Constant solutions to an autonomous differential equation or system of DEs are called equilibrium solutions. Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system

$$F(x, y) = 0$$
$$G(x, y) = 0$$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems

$$x' = F(x, y)$$
$$y' = G(x, y)$$

are called stable if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.

- Equilibrium solutions are unstable if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called asymptotically stable if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.

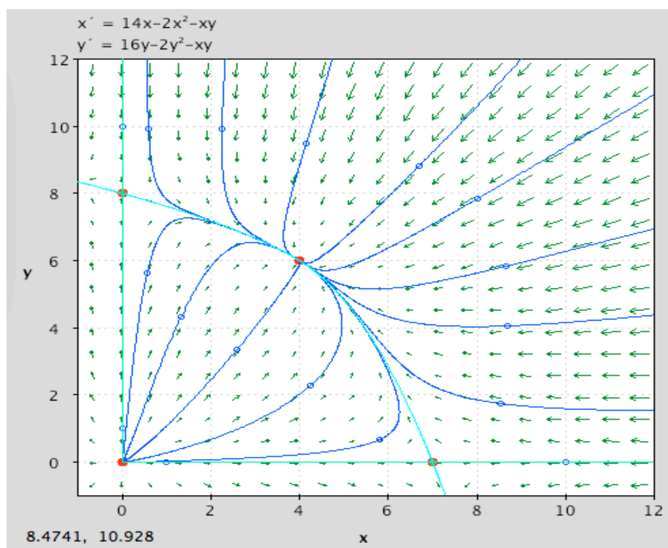
(Notice these definitions are completely analogous to our discussion in Chapter 2.)

Example 1) Consider the "competing species" model from section 6.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y\,.$$

1a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other). Hint: to understand why this model is plausible for $x(t)$ consider the normalized birth rate rate $\dfrac{x'(t)}{x(t)}$, as we did in Chapter 2.

1b) Find the four equilibrium solutions to this competition model, algebraically.
1c) What does the phase portrait below indicate about the dynamics of this system?
1d) Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability, based on what the phase portrait looks like near each equilibrium solution?



Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math 2280 course. (You may have forgotten, but the "linear drag" velocity model, Newton's law of cooling, small oscillation pendulum motion, and the damped spring equation were all linearizations!!) It's important to understand how to linearize in general, because the linearized differential equations can often be used to understand stability and solution behavior near the equilibrium point, for the original differential equations. Today we'll talk about linearizing systems of DE's, which we've not done before in this class.

An easy case of linearization in Example 1 is near the equilbrium solution $[x_*, y_*]^T = [0, 0]^T$. It's pretty clear that our population system

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

linearizes to

$$x'(t) = 14x$$
$$y'(t) = 16y$$

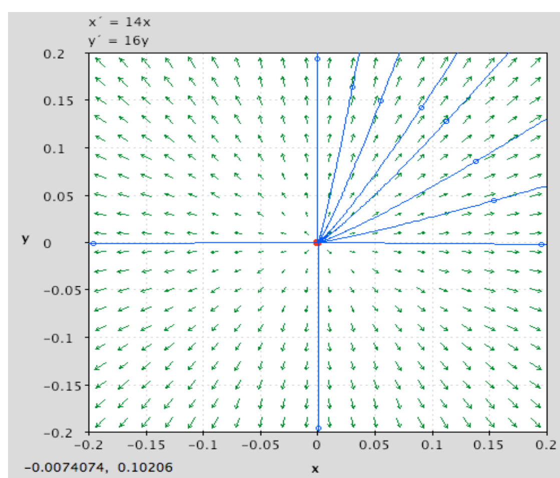
i.e.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:



After the break, we'll talk about how to linearize with multivariable Calculus at equilibrium points that are not the origin. (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter.) If you want to read ahead, here's the idea:

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \ .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \ .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*)\, u + \frac{\partial F}{\partial y}(x_*, y_*)\, v + \varepsilon_1(u, v)$$
$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*)\, u + \frac{\partial G}{\partial y}(x_*, y_*)\, v + \varepsilon_2(u, v)$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \rightarrow 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

The matrix of partial derivatives is called the Jacobian matrix for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Example 2) We will linearize the rabbit-squirrel (competition) model of the previous example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

2a) Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

2b) The matrix in the linear system of DE's above has approximate eigendata:

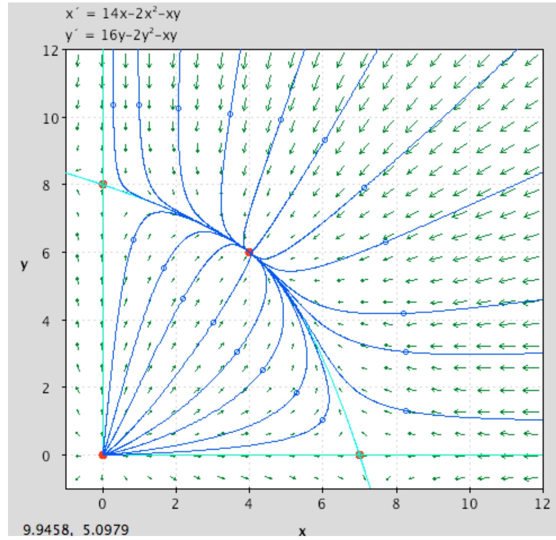
$$\lambda_1 \approx -4.7, \quad \underline{\mathbf{v}}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{\mathbf{v}}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$. This is sensible, since the correspondence between (x, y) and (u, v) involves a translation of $x - y$ coordinate axes to $u - v$ coordinate axes, via the formula.

$$x = u + 4$$
$$y = v + 6$$

Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

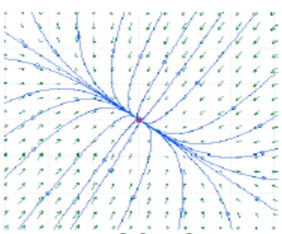
The non-linear problem and representative solution curves:



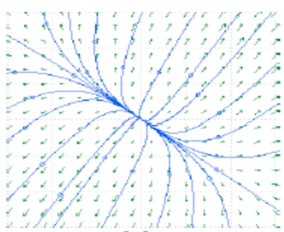
pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

```
Equilbrium Point:
    There is a nodal sink at (4, 6)
    Jacobian:
          -8        -4
          -6        -12
    The eigenvalues and eigenvectors are:
          -4.7085    (0.77218, -0.63541)
          -15.292    (0.48097, 0.87674)
```

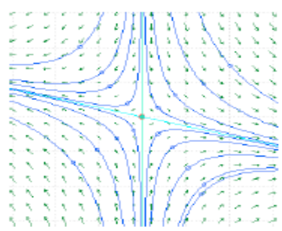
The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



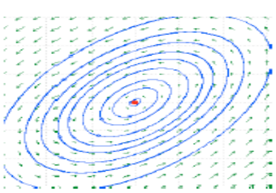
nodal sink
$\lambda_1, \lambda_2 < 0$



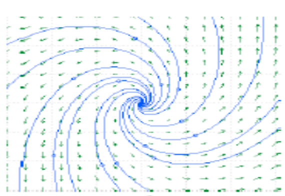
nodal source
$\lambda_1, \lambda_2 > 0$



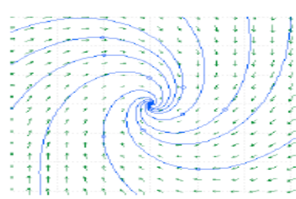
saddle point
$\lambda_1 < 0 < \lambda_2$



center
$Re(\lambda)=0$



spiral source
$Re(\lambda)>0$



spiral sink
$Re(\lambda)<0$

Theorem: Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.

(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.

(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Phase potrait for undamped, freely rotating rigid-rod pendulum differential equation with $\frac{g}{L} = 1$,

$$\theta''(t) + \sin(\theta(t)) = 0$$

converted into its equivalent first order autonomous system of differential equations. We shall return to this after the break, and also add damping.