

Math 2280-002
Week 8: 3.6 -3.7, 4.1-4.2
Mon Feb 25
3.6: Forced oscillations

Announcements:

today & tomorrow : study important
physics phenomena related to forced oscillation
problems (3.6)

Warm-up Exercise: What is your undetermined coefficients guess for $x_p(t)$?

a) $x''(t) + 4x(t) = 5\cos 3t$

$x_p(t) = d_1 \cos 3t$
(don't need $\sin 3t$
terms since only
even derivs in L)

b) $x''(t) + 4x(t) = 5\cos 2t$

$x_H(t) = \text{span}\{\cos 2t, \sin 2t\}$

$x_p(t) = d_1 t \cos 2t + d_2 t \sin 2t$

c) $x''(t) + 2x'(t) + 4x(t) = 5\cos 2t$

$x_p(t) = d_1 \cos 2t + d_2 \sin 2t$

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:

We study solutions $x(t)$ to

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped ($c = 0$) :

In this case the complementary homogeneous differential equation for $x(t)$ is

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0$$

example
on Friday

which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)$$

So for the non-homogeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A \cos(\omega t)$ because only even derivatives, we don't need

$\sin(\omega t)$ terms !!

$$\Rightarrow x = x_P + x_H = A \cos(\omega t) + C_0 \cos(\omega_0 t - \alpha)$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0, A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

interesting phenomenon

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

x

our undetermined coefficients guess is

$$\begin{aligned} x_P &= t^1 (A \cos(\omega_0 t) + B \sin(\omega_0 t)) \\ \Rightarrow x &= x_P + x_H = C t \cos(\omega t - \alpha) + C_0 \cos(\omega_0 t - \alpha) \end{aligned}$$

("pure" resonance!)

linearly growing amplitude
interesting phenomenon.

part (b) on warmup.

- damped ($c > 0$): in all cases $x_P = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$ (because the roots of the characteristic polynomial are never purely imaginary $\pm i \omega$ when $c > 0$).

- underdamped: $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1)$
- critically-damped: $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} (c_1 t + c_2)$
- over-damped: $x = x_P + x_H = C \cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$

part (c) warmup

- in all three damped cases on the previous page, $x_H(t) \rightarrow 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \rightarrow \infty$).

$x_p(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \rightarrow \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

3rd interesting phenomenon

We worked this Exercise on Friday...

Forced undamped oscillations: (We'll discuss forced damped oscillations on Monday next week.)

Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80 \cos(5t)$$

$$x(0) = 0$$

$$x'(0) = 0.$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

soln from warmup

$$x(t) = -5 \cos 5t + 5 \cos 3t$$

$\cos \omega t$
 $\omega = \text{angular freq } \frac{\text{rad}}{\text{time}}$
 $f = \frac{\omega}{2\pi} \quad \text{cycles/time}$
 $T = \frac{2\pi}{\omega} \quad \text{time/cycle}$

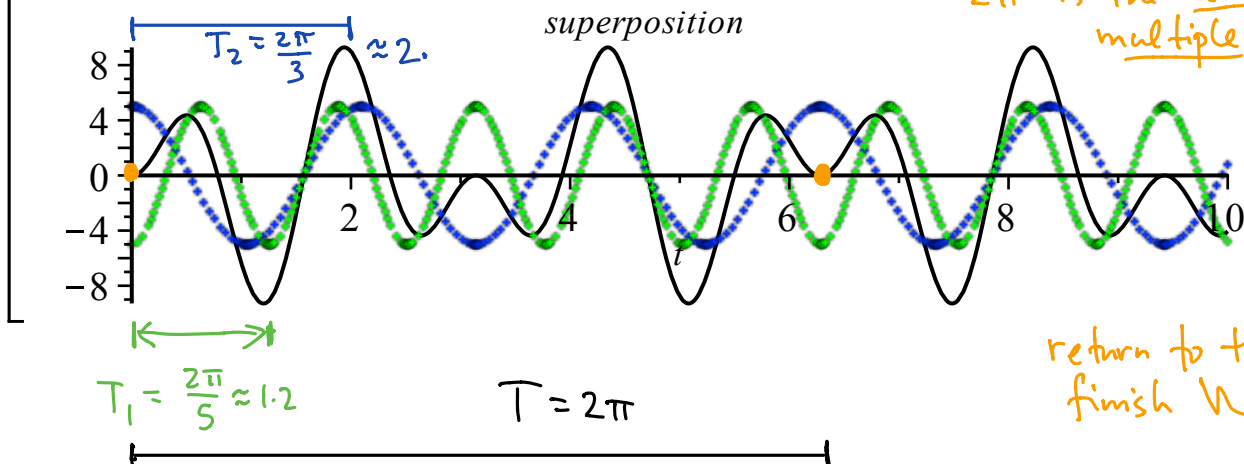
$\text{period } T_1 = \frac{2\pi}{5}$
 $\cos 5\left(\frac{2\pi}{5}\right) = \cos 2\pi = \cos 0$
 $T_2 = \frac{2\pi}{3}$

the period of the sum
 is (well it looks to be) 2π
 2π is the least common multiple of T_1 & T_2

$2\pi = 5T_1$
 $2\pi = 3T_2$

```

> with(plots):
> plot1 := plot(-5*cos(5*t), t=0..10, color=green, style=point):
> plot2 := plot(5*cos(3*t), t=0..10, color=blue, style=point):
> plot3 := plot(-5*cos(5*t) + 5*cos(3*t), t=0..10, color=black):
> display({plot1, plot2, plot3}, title='superposition');
  
```



return to this after
finish Wed notes

In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the

case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$:

no damping.

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m} \cos(\omega t)$$

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m} \cos \omega t$$

$$x_H(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$x(0) = x_0$$

$$x'(0) = v_0$$

//
 $L(x(t))$

$$x_p(t) = d \cos \omega t$$

$$L(x_p(t)) = d L(\cos \omega t)$$

$$= d (\cos \omega t) [-\omega^2 + \omega_0^2] \stackrel{\text{want}}{=} \frac{F_0}{m} \cos \omega t$$

$$d(\omega_0^2 - \omega^2) = \frac{F_0}{m}$$

$$\Rightarrow d = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$$x = x_p + x_H$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$x(0) = x_0 = \frac{F_0}{m(\omega_0^2 - \omega^2)} + c_1 \Rightarrow c_1 = x_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x'(0) = v_0 = c_2 \omega_0 \Rightarrow c_2 = \frac{v_0}{\omega_0}$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + \left[x_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)} \right] \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

Solution:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos(\omega_0 t) - \cos(\omega t)) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$- (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))$$

$$= 2 \sin(\alpha)\sin(\beta)$$

$$\alpha - \beta = \omega_0 t \quad / \quad \alpha + \beta = \omega t$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \left(2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) \right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \quad \text{period: } \frac{4\pi}{|\omega - \omega_0|}$$

We will call half that period the beating period, as explained by the next exercise:

beating period: $\frac{2\pi}{|\omega - \omega_0|}$, beating amplitude: $\frac{2F_0}{m|\omega^2 - \omega_0^2|}$.

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos(\omega_0 t) - \cos(\omega t)) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t).$$

Exercise 2a) Use one of the formulas above to write down the IVP solution $x(t)$ to

$$\begin{aligned} x'' + 9x &= 80 \cos(3.1t) \\ x(0) &= 0 \\ x'(0) &= 0. \end{aligned}$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

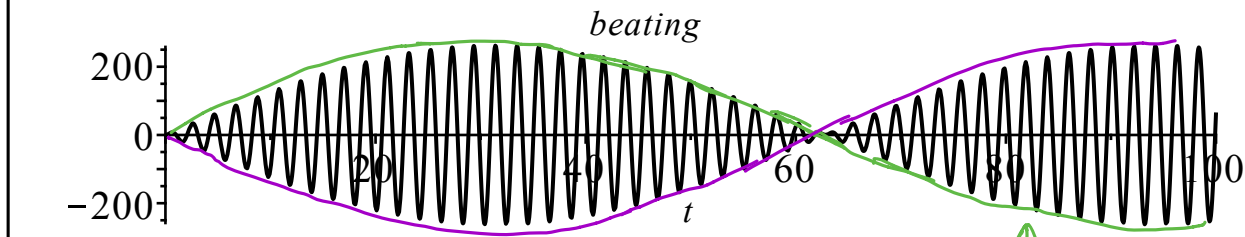
$$\begin{aligned} \frac{F_0}{m} &= 80 \\ \omega_0 &= 3 \\ \omega &= 3.1 \\ x_0 = v_0 &= 0 \\ \frac{\omega - \omega_0}{2} &= .05 \end{aligned}$$

$$x(t) = \frac{80}{\underbrace{(3.1)^2 - 3^2}_{262.3}} 2 \sin(3.05t) \sin(.05t) + 0 + 0$$

period $\frac{2\pi}{.05} = 40\pi \approx 124$

beating period
= 20π

> plot(262.3 * sin(3.05 * t) * sin(.05 * t), t = 0 .. 100, color = black, title = 'beating');



oscillations between
 $\pm 262.3 \sin(.05t)$
are caused by $\sin(3.05t)$

$$x = 262.3 \sin(.05t)$$

Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \rightarrow \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using 5.5, guess

$$\begin{aligned} + \omega_0^2 (& x_p = t (A \cos \omega_0 t + B \sin \omega_0 t)) \\ 0 (& x_p' = t (-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t) + A \cos \omega_0 t + B \sin \omega_0 t) \\ + 1 (& x_p'' = t (-A \omega_0^2 \cos \omega_0 t - B \omega_0^2 \sin \omega_0 t) + [-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t] 2) \end{aligned}$$

$$L(x_p) = t(0) + 2[-A \omega_0 \sin \omega_0 t + B \omega_0 \cos \omega_0 t] \stackrel{\text{want}}{=} \frac{F_0}{m} \cos \omega_0 t$$

$$\text{Deduce } A = 0 \\ B = \frac{F_0}{2m\omega_0}$$

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

sats $x(0)=0$
 $x'(0)=0$, so IVP soln is

$$x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

You can also get this solution by letting $\omega \rightarrow \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$\begin{aligned}x''(t) + \frac{k}{m}x(t) &= \frac{F_0}{m}\cos(\omega t) \\x(0) &= x_0 \\x'(0) &= v_0\end{aligned}$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} (\cos(\omega_0 t) - \cos(\omega t)) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) .$$

If we let $\omega \rightarrow \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

$$x(t) = \frac{F_0}{m(\omega + \omega_0)(\omega - \omega_0)} \sin\left(\frac{\omega + \omega_0}{2}t\right) \sin\left(\frac{\omega - \omega_0}{2}t\right) \left(\frac{\omega - \omega_0}{2}t\right) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

$$x_p(t) \rightarrow \frac{2F_0}{2m\omega_0} (\sin \omega_0 t) \frac{t}{2} \quad \omega \rightarrow \omega_0 \quad \text{on any interval } [0, M] \quad \downarrow 1 \quad \frac{\sin \theta}{\theta} \rightarrow 1 \quad \text{as } \theta \rightarrow 0$$

so in the limit,

$$x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

Resonance summary:

$$\begin{aligned}x''(t) + \omega_0^2 x(t) &= \frac{F_0}{m} \cos(\omega_0 t) \\x(0) &= x_0 \\x'(0) &= v_0\end{aligned}$$

has solution

$$\bullet \quad x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

Exercise 3a) Solve the IVP

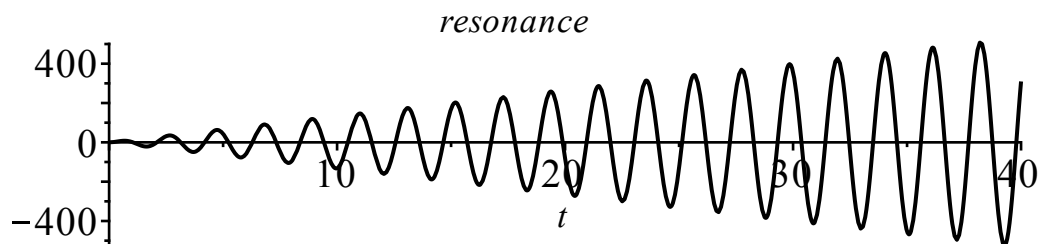
$$\begin{aligned}x'' + 9x &= 80 \cos(3t) \\x(0) &= 0 \\x'(0) &= 0.\end{aligned}$$

Just use the general solution formula above this exercise and substitute in the appropriate values for the various terms.

$$x(t) = \frac{40}{3} t \sin 3t$$

3b) Compare the solution graph below with the beating graph in exercise 2.

```
> plot( (40/3) * t * sin(3 * t), t = 0..40, color = black, title = `resonance` );
```



```
>
```

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)

http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)

http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)

<http://www.youtube.com/watch?v=j-zczJXSxw> (Tacoma narrows bridge)

http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)

http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Tues Feb 26

3.6-3.7: Forced oscillations: practical resonance; 3.7 Electrical circuits analog

Announcements:

Studying yesterday $\left\{ \begin{array}{l} \text{beating } \omega \approx \omega_0 \\ \text{resonance } \omega = \omega_0 \end{array} \right. \quad x''(t) + \omega_0^2 x(t) = \frac{F_0}{m} \cos \omega t$

today: practical resonance $x''(t) + c x' + \omega_0^2 x$

$\left\{ \begin{array}{l} \text{when } c \text{ "small"} \\ \omega \approx \omega_0 \end{array} \right. = \frac{F_0}{m} \cos \omega t$

Homework questions?

Warm-up Exercise:

Find a particular solution $x(t)$ to

$$x'' + 2x' + 26x = 82 \cos 4t$$

$$26(x_p = d_1 \cos 4t + d_2 \sin 4t)$$

$$+ 2(x_p' = -4d_1 \sin 4t + 4d_2 \cos 4t)$$

$$+ 1(x_p'' = -16d_1 \cos 4t - 16d_2 \sin 4t)$$

$$L(x_p) = \cos 4t (26d_1 + 8d_2 - 16d_1)$$

$$+ \sin 4t (26d_2 - 8d_1 - 16d_2)$$

want $\begin{aligned} &= \cos 4t (82) \\ &+ \sin 4t (0) \end{aligned}$

$$10d_1 + 8d_2 = 82$$

$$-8d_1 + 10d_2 = 0$$

$\div 2$

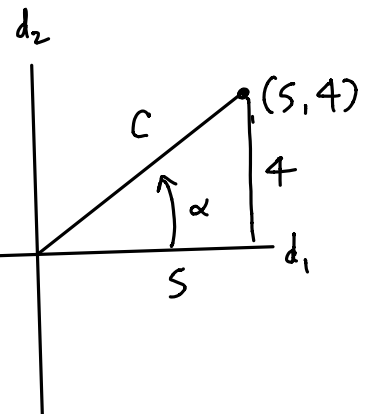
$$\begin{bmatrix} 5 & 4 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 5 & -4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 41 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$x_p(t) = 5 \cos 4t + 4 \sin 4t$$

$$= C \cos(4t - \alpha)$$

$$= \sqrt{41} \cos(4t - \alpha)$$

$$\alpha = \arctan\left(\frac{4}{5}\right)$$



Damped forced oscillations ($c > 0$) for $x(t)$:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_p(t)$:

$$\begin{aligned} & k [x_p = A \cos(\omega t) + B \sin(\omega t)] \\ & + c [x_p' = -A \omega \sin(\omega t) + B \omega \cos(\omega t)] \\ & + m [x_p'' = -A \omega^2 \cos(\omega t) - B \omega^2 \sin(\omega t)] . \end{aligned}$$

$$\frac{\quad}{L(x_p) = \cos(\omega t) (kA + cB\omega - mA\omega^2) + \sin(\omega t) (kB - cA\omega - mB\omega^2)} \quad \begin{array}{l} \text{want} \\ \cos \omega t (F_0) \\ + \sin \omega t (0) \end{array}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 \\ c\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_p = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad (\text{Check!})$$

$$\cos(\alpha) = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \rightarrow 0 < \alpha < \pi$$

And the general solution $x(t) = x_p(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + e^{-p t} (c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

From previous page:

$$x_p = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$$

Since

$$k - m\omega^2 = m(\omega_0^2 - \omega^2)$$

we may rewrite the steady periodic trig data as

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{c^2}{m^2}\omega^2}} .$$

$$\cos(\alpha) = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{c^2}{m^2}\omega^2}}$$

$$\sin(\alpha) = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{c\omega}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{c^2}{m^2}\omega^2}} .$$

$$k - m\omega^2 = m(\omega_0^2 - \omega^2) .$$

Exercise 1) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c ! Namely, for c small, when $\omega^2 < \omega_0^2$ we have $\cos(\alpha) \approx 1$, i.e. $\alpha \approx 0$ (in phase with the forcing function) for x_{sp} ; when $\omega^2 > \omega_0^2$ we have $\cos(\alpha) \approx -1$, i.e. α near π (out of phase with the forcing function); for $\omega \approx \omega_0$, $\sin(\alpha) \approx 1$, i.e. $\alpha \approx \frac{\pi}{2}$.

<http://www.youtube.com/watch?v=aZNnwQ8HJHU>

Exercise 2) Find $x_{sp}(t)$ for the forced oscillation problem

$$\begin{aligned} x'' + 2x' + 26x &= 82 \cos(4t) \\ x(0) &= 6 \\ x'(0) &= 0 . \end{aligned}$$

did this!

Practical resonance: The steady periodic amplitude C for damped forced oscillations is

$$C(\omega) = C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{c^2}{m^2}\omega^2}}.$$

Notice that as $\omega \rightarrow 0$, $C(\omega) \rightarrow \frac{F_0}{k} = \frac{F_0}{m\omega_0^2}$ and that as $\omega \rightarrow \infty$, $C(\omega) \rightarrow 0$. The precise definition of

practical resonance occurring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval

$0 < \omega < \infty$. (Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in the variable ω^2 it will have at most one minimum in the variable ω^2 , so $C(\omega)$ will have at most one maximum for non-negative ω . It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 3a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c :

$$x'' + cx' + 26x = 82 \cos(\omega t).$$

3b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$$C(\omega) = 82 \frac{1}{\sqrt{(26 - \omega^2)^2 + 4\omega^2}}$$

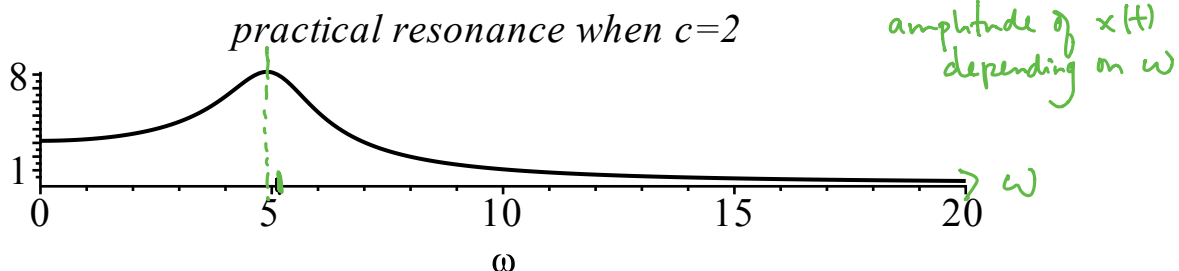
> restart :

> with(plots) :

> $C := (\omega, c) \rightarrow \frac{82}{\sqrt{(26 - \omega^2)^2 + c^2 \cdot \omega^2}} :$

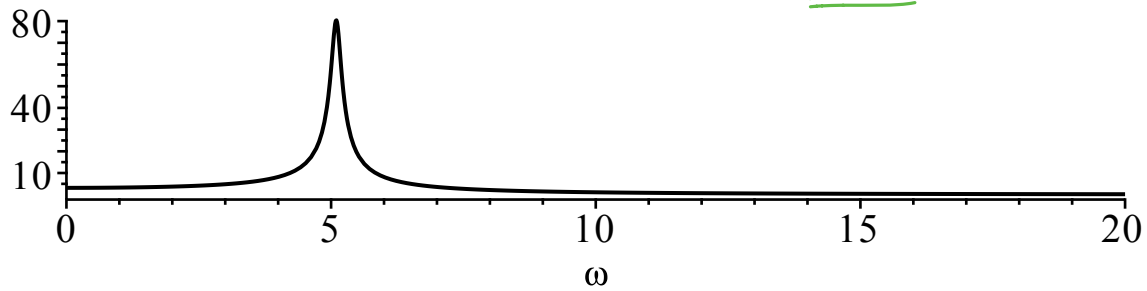
$$\omega_0 = \sqrt{26} \approx 5$$

> plot($C(\omega, 2)$, $\omega = 0..20$, color = black, title = 'practical resonance when $c=2$ ');



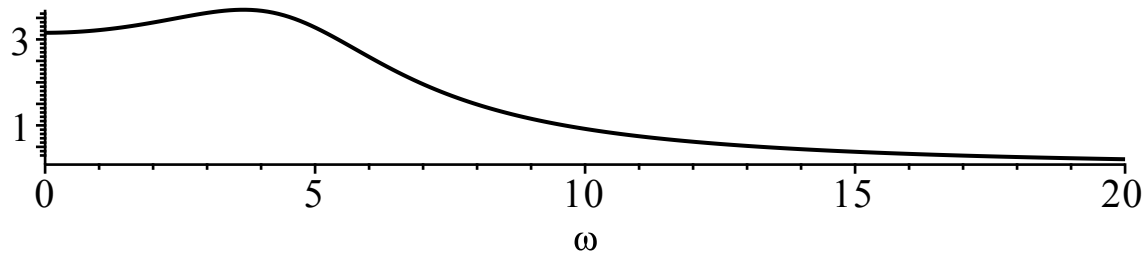
> `plot(C(ω , .2), ω = 0 ..20, color = black, title = `serious practical resonance when $c=0.2`$);`

serious practical resonance when $c=0.2$



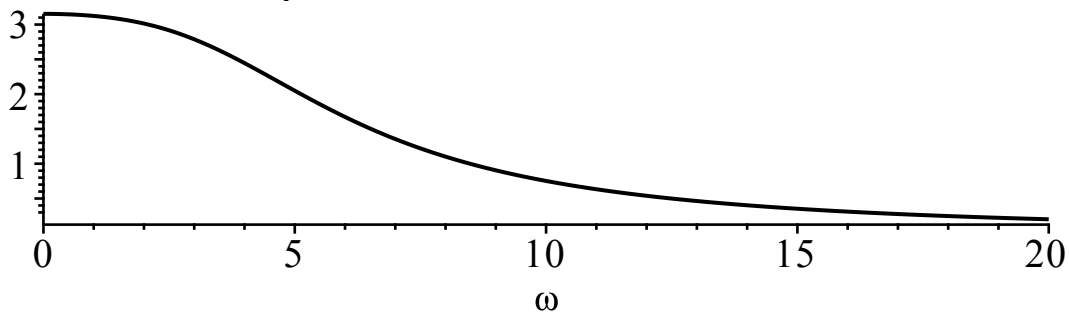
> `plot(C(ω , 5), ω = 0 ..20, color = black, title = `barely practical resonance when $c=5`$);`

barely practical resonance when $c=5$

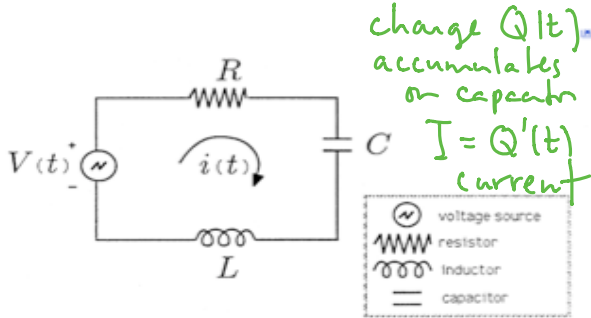


> `plot(C(ω , 8), ω = 0 ..20, color = black, title = `no practical resonance when $c=8`$);`

no practical resonance when $c=8$



Section 3.7 The mechanical-electrical analogy: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....the classical RLC circuit with applied voltage is described in this schematic:



circuit element	voltage drop	units
inductor	$L I'(t)$	L Henries (H)
resistor	$R I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C} Q(t)$	C Farads (F)

<http://cnx.org/content/m21475/latest/pic012.png>

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $L Q''(t) + R Q'(t) + \frac{1}{C} Q(t) = V(t) = E_0 \sin(\omega t)$ volts $= 120 \sin(60 \cdot 2\pi t)$
 For $I(t)$: $L I''(t) + R I'(t) + \frac{1}{C} I(t) = V'(t) = E_0 \omega \cos(\omega t)$ 60 Herz (cycles/sec)

We can transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t).$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma), \quad \gamma = \alpha - \frac{\pi}{2}.$$

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2}}.$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the *impedance*, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\omega} = L\omega$, i.e.

$$C = \frac{1}{L\omega^2} \text{ if } L \text{ is fixed and } C \text{ is adjustable (old analog radios).}$$

$$L = \frac{1}{C\omega^2} \text{ if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:

http://www.youtube.com/watch?v=ZYgFuUI9_Vs.

Wed Feb 27

4.1 Introduction to systems of differential equations

• Chapters 4–6 systems of differential eqns

Announcements:

- I pointed out on next HW when we'll talk about related examples

- Quiz today on § 3.6 : Forced oscillations
three important physical phenomena

- beating
- resonance (pure)
- practical resonance

$$m x'' + c x' + k x = F_0 \cos \omega t$$

Warm-up Exercise: !! Find the eigenvalues and eigenvectors (eigenspace bases) for

$$A = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$\lambda = 0, -6$$

$$E_{\lambda=0} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

✓ check $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
($A\vec{v} = \lambda\vec{v}$)

$$E_{\lambda=-6} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

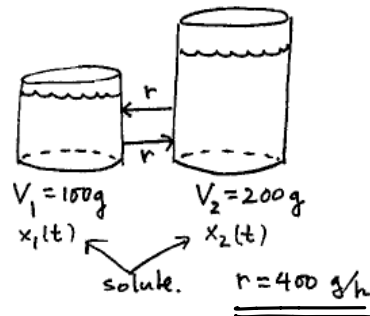
✓ check $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model

and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t)$, $x_2(t)$ above, using input-output modeling.

Assume solute concentration is uniform in each tank. If $x_1(0) = b_1$, $x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r_i c_i - r_o c_o \quad r_i = r_o = 400$$

$$x_1'(t) = 400 \frac{x_2}{200} - 400 \frac{x_1}{100} = -4x_1 + 2x_2$$

$\frac{\text{gal}}{\text{hour}} \quad \frac{\text{lb}}{\text{gal}}$

$$x_2'(t) = r_i c_i - r_o c_o = 400 \frac{x_1}{100} - 400 \frac{x_2}{200}$$

$$x_2'(t) = 4x_1 - 2x_2$$

A in warmup.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1'(t) = -4x_1 + 2x_2$$

$$x_2'(t) = 4x_1 - 2x_2$$

$$x_1(0) = b_1$$

$$x_2(0) = b_2$$

The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\text{IVP} \left\{ \begin{array}{l} \mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{array} \right.$$

$$\vec{x}(t): \vec{x}: [a, b] \rightarrow \mathbb{R}^n$$

Chapter 1

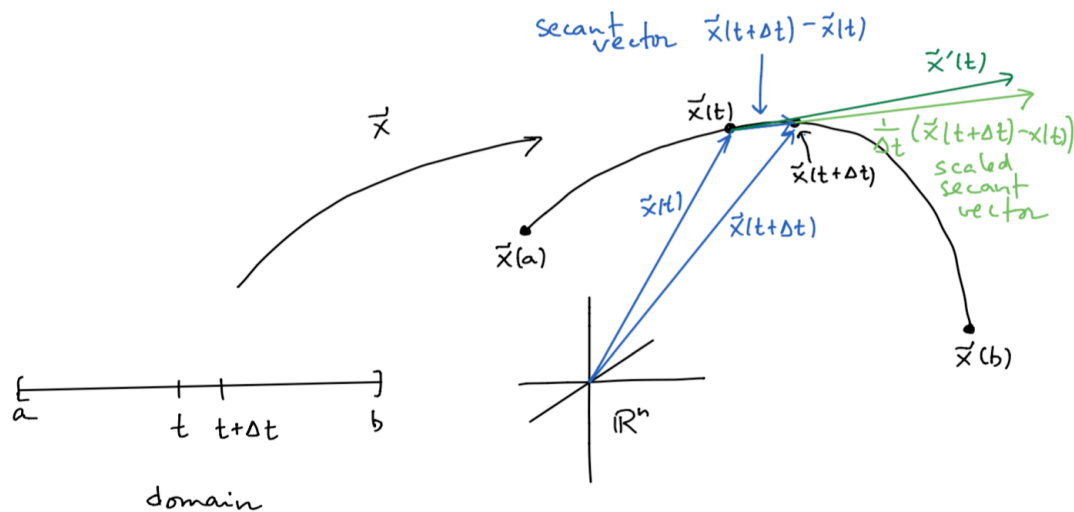
$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

Existence and Uniqueness for solutions to IVP's: The IVP above is a vectorized version of the scalar first order DE IVP that we considered in Chapter 1. In Chapter 1 we understood why (with the right conditions on the right hand side), these IVP's have unique solutions. There is an analogous existence-uniqueness theorem for the vectorized version we study in Chapters 4-6, and it's believable for the same reasons the Chapter 1 theorem seemed reasonable. We just have to remember the geometric meaning of the *tangent* vector $\mathbf{x}'(t)$ to a parametric curve in \mathbb{R}^n (which is also called the *velocity* vector in physics, when you study particle motion):

Algebra:

$$\mathbf{x}'(t) := \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t + \Delta t) \\ x_2(t + \Delta t) \\ \vdots \\ x_n(t + \Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right) = \lim_{\Delta t \rightarrow 0} \begin{bmatrix} \frac{1}{\Delta t} (x_1(t + \Delta t) - x_1(t)) \\ \frac{1}{\Delta t} (x_2(t + \Delta t) - x_2(t)) \\ \vdots \\ \frac{1}{\Delta t} (x_n(t + \Delta t) - x_n(t)) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

Geometric interpretation in terms of displacement vectors along a parametric curve:



So the existence-uniqueness theorem for first order systems of DE's is true because if you know where you start at time t_0 , namely \mathbf{x}_0 ; and if you know your tangent vector $\mathbf{x}'(t)$ at every later time -in terms of your location $\mathbf{x}(t)$ and what time t it is, as specified by the vector function $\mathbf{F}(t, \mathbf{x}(t))$; then there should only be one way the parametric curve $\mathbf{x}(t)$ can develop. This is analogous to our reasoning in Chapter 1 that there should only be one way to follow a slope field, given the initial point one starts at.

Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4x_1 + 2x_2$$

$$x_2'(t) = 4x_1 - 2x_2$$

$$x_1(0) = 9 \quad \leftarrow \text{all solute starts in tank 1}$$

$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" (tangent vector) vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time t .

If your system involved ten coupled tanks rather than two, then this "particle" is moving around in \mathbb{R}^{10} .

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?

$$2b) \lim_{t \rightarrow \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2c) because total salt is 9 lb
& eventually concentration should
be equal everywhere

& 2nd tank
is twice
as large

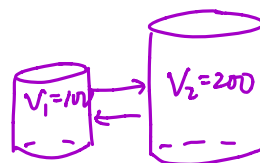
$$= (4x_1 - 2x_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 2(2x_1 - x_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

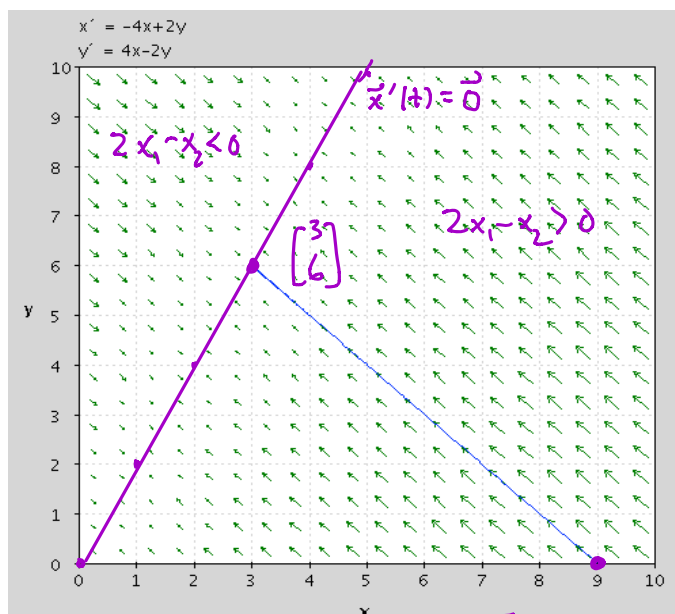
$$\vec{x}'(t) = \vec{0}$$

$$\text{when } 2x_1 - x_2 = 0$$

$$x_2 = 2x_1$$



pplane $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix}$



$$\begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Friday starts here

Definition: Any first order system of differential equations which can be written in the form

Chptr 1

$$\mathbf{x}'(t) + P(t) \mathbf{x} = \mathbf{f}(t)$$

$$\mathbf{x}'(t) + P(t) \mathbf{x}(t) = \mathbf{f}(t)$$

is called a *first order linear system of DE's*. (Here $\mathbf{x}(t)$ and $\mathbf{f}(t)$ are functions from an interval in \mathbb{R} , with range lying in \mathbb{R}^n , and $P(t)$ is an $n \times n$ matrix whose entries are functions of t . For us $P(t)$ will almost always be a constant matrix. If the system can be written in the form

$$\mathbf{x}'(t) + P(t) \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}'(t) + P(t) \mathbf{x}(t) = \mathbf{0}$$

we say that the linear system of differential equations is *homogeneous*. Otherwise it is *non-homogeneous* or *inhomogeneous*.

Notice that the operator on vector-valued functions $\mathbf{x}(t)$ defined by

$$L(\mathbf{x}(t)) := \mathbf{x}'(t) + P(t) \mathbf{x}(t)$$

is linear, i.e.

$$L(\mathbf{x}(t) + \mathbf{y}(t)) = L(\mathbf{x}(t)) + L(\mathbf{y}(t))$$

$$L(c \mathbf{x}(t)) = c L(\mathbf{x}(t)).$$

$$\begin{aligned} L(\vec{x}(t) + \vec{y}(t)) &= (\vec{x}(t) + \vec{y}(t))' \\ &\quad + P(t) [\vec{x}(t) + \vec{y}(t)] \\ &= \vec{x}' + \vec{y}' \\ &\quad + P(t) \vec{x} + P(t) \vec{y} \\ &= L(\vec{x}(t)) + L(\vec{y}(t)) \end{aligned}$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\mathbf{x}'(t) + P(t) \mathbf{x} = \mathbf{0}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\mathbf{x}'(t) + P(t) \mathbf{x} = \mathbf{f}(t)$$

will be of the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_H$$

where \mathbf{x}_p is any single particular solution and \mathbf{x}_H is the general homogeneous solution.

In the case that $P(t) = -A$ is a constant matrix (i.e. entries don't depend on t), we usually write the homogeneous system as

$$\mathbf{x}'(t) = A \mathbf{x}.$$

e.g. tank example.

In the case that A is a diagonalizable matrix it turns out we can always find a basis for the homogeneous solution space made of vector-valued functions of the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{y},$$

where \mathbf{y} an eigenvector of A and λ is its eigenvalue, i.e.

$$\underline{A \mathbf{y} = \lambda \mathbf{y}.}$$

System of DE's:

$$\mathbf{x}'(t) = A \mathbf{x}$$

Candidate solution:

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v},$$

where \mathbf{v} an eigenvector of A and λ is its eigenvalue, i.e.

$$A \mathbf{v} = \lambda \mathbf{v}.$$

We can verify that such an $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ solves the homogeneous DE system above by showing we get a true identity when we substitute it in. We compute the left side of the differential equation:

LHS

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v} \Rightarrow \mathbf{x}'(t) = \lambda e^{\lambda t} \mathbf{v}.$$

And we compute the right side

RHS

$$A \mathbf{x}(t) = A e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v} = e^{\lambda t} \lambda \mathbf{v}.$$

Same!

$$\frac{d}{dt} \begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \vdots \\ e^{\lambda t} v_n \end{bmatrix} = \begin{bmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \\ \vdots \\ \lambda e^{\lambda t} v_n \end{bmatrix} = \lambda e^{\lambda t} \mathbf{v} \quad \checkmark$$

Exercise 3) Use the eigendata of the matrix in our running example solve the initial value problem of Exercise 2!! Compare your solution $\vec{x}(t)$ to the parametric curve drawn by pplane.

① eigendata for A $(A\vec{v} = \lambda\vec{v} \Leftrightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \Leftrightarrow (A - \lambda I)\vec{v} = \vec{0})$ non-zero sol's iff $A - \lambda I$ is not invertible $(A - \lambda I = 0)$

$$|A - \lambda I| = \begin{vmatrix} -4 - \lambda & 2 \\ 4 & -2 - \lambda \end{vmatrix} = (\lambda + 4)(\lambda + 2) - 8 = \lambda^2 + 6\lambda - 8$$

$$((-4 - \lambda)(-2 - \lambda) - 8 = (-1)(-1)(\lambda + 4)(\lambda + 2))$$

$$|A - \lambda I| = \lambda(\lambda + 6) = 0$$

roots $\lambda = 0, -6$, are eigenvalues

$$E_{\lambda=0} = \text{Nul}(A - 0I) \quad (\text{or kernel})$$

↑
eigenspace
 $\lambda=0$

$$\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}$$

$$1 \cdot \text{col}_1 + 2 \cdot \text{col}_2 = \vec{0}$$

$$E_{\lambda=0} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

OR reduce matrix.

$$R_1 \rightarrow \frac{R_1}{-4}: \begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \\ \hline 1 & -\frac{1}{2} & 0 \end{array}$$

$$-4R_1 + R_2 \rightarrow R_2 \quad \begin{array}{cc|c} 0 & 0 & 0 \end{array}$$

$$\text{backsolve: } v_1 = \frac{1}{2}v_2, \quad v_2 = \text{free}$$

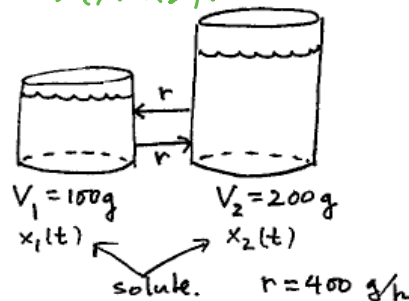
$$\vec{v} = v_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$E_{\lambda=0} = \text{span} \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$E_{\lambda=-6} = \text{Nul}(A - (-6)I)$$

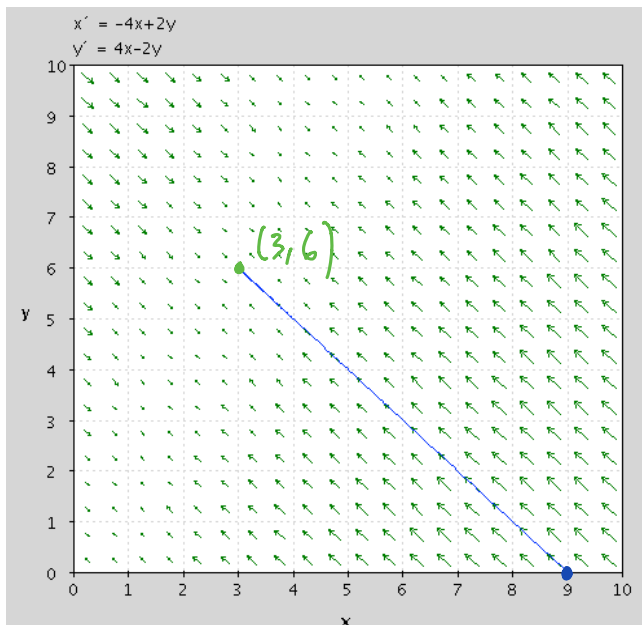
$$\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array} \rightarrow \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$E_{\lambda=-6} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$



$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



solutions to $\vec{x}' = A\vec{x}$ $e^{\lambda t} \vec{v}$: $e^{0t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

solutions $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

because homog. linear DE

IVP: $\vec{x}(0) = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \stackrel{\text{want}}{=} c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Fri Mar 1

5.1-5.2 Systems of differential equations and the vector Calculus we need to study them (5.1). Every differential equation or system of differential equations can be converted into a first order system of differential equations (4.1).

Announcements:

continue from yesterday ...

$$\text{Soln } \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

consistent with graph on previous page

- check: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3+6 \\ 6-6 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \checkmark$
- $\lim_{t \rightarrow \infty} \vec{x}(t) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \checkmark$
- $\vec{x}'(t) = -36e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 36e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \checkmark$

Warm-up Exercise: For the first order system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(from tank problem Wed.)

Show that

$$\vec{z}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{z}_2(t) = e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{are solutions}$$

plug them into the DE system

$$\text{LHS: } \vec{z}_1'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$e^{0t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{RHS } \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

LHS = RHS ... $\vec{z}_1(t)$ is a soln.

$$\text{LHS: } \vec{z}_2'(t) = -6e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{(check: } \frac{d}{dt} \begin{bmatrix} -1e^{-6t} \\ e^{-6t} \end{bmatrix} = \begin{bmatrix} 6e^{-6t} \\ -6e^{-6t} \end{bmatrix})$$

same

$$\text{RHS: } \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-6t} \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

LHS = RHS

... $\vec{z}_2(t)$ is a soln.

$$= e^{-6t} \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

On Wednesday, we began and maybe finished solving the two-tank IVP example analytically, using eigenvalues and eigenvectors from the matrix A in that problem (!). That is typical of what we will do in section 5.2, to solve the first order system of DE's

$$\begin{aligned}\mathbf{x}'(t) &= A \mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0.\end{aligned}$$

At this point though, it's a good idea to review and extend some differentiation rules you probably learned in multivariable Calculus, when you studied the calculus of parametric curves. This is related to material in section 5.1 of the text, e.g. page 271. Most of the rest of section 5.1 is material you learned in Math 2270 - you may wish to scan it to make sure it's still familiar.

1) If $\mathbf{x}(t) = \mathbf{b}$ is a constant vector, then $\mathbf{x}'(t) = \mathbf{0}$ for all t , and vice-versa. (Because all of the entries in the vector \mathbf{b} are constants, and their derivatives are zero. And if the derivatives of all entries of a vector are identically zero, then the entries are constants.)

2) Sum rule for differentiation:

$$\frac{d}{dt}(\mathbf{x}(t) + \mathbf{y}(t)) = \mathbf{x}'(t) + \mathbf{y}'(t): \quad \text{Both sides simplify to} \quad \begin{bmatrix} x_1'(t) + y_1'(t) \\ x_2'(t) + y_2'(t) \\ \vdots \\ x_n'(t) + y_n'(t) \end{bmatrix}$$

3) Constant multiple rule for differentiation:

$$\frac{d}{dt}(c \mathbf{x}(t)) = c \mathbf{x}'(t): \quad \text{Both sides simplify to} \quad c \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

4) Matrix-valued functions sometimes show up and sometimes need to be differentiated. This is done with the limit definition, and amounts to differentiating each entry of the matrix. For example, if $A(t)$ is a 2×2 matrix, then

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{bmatrix} a_{11}(t + \Delta t) & a_{12}(t + \Delta t) \\ a_{21}(t + \Delta t) & a_{22}(t + \Delta t) \end{bmatrix} - \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \right) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \begin{bmatrix} a_{11}(t + \Delta t) - a_{11}(t) & a_{12}(t + \Delta t) - a_{12}(t) \\ a_{21}(t + \Delta t) - a_{21}(t) & a_{22}(t + \Delta t) - a_{22}(t) \end{bmatrix} \\ &= \lim_{\Delta t \rightarrow 0} \begin{bmatrix} \frac{a_{11}(t + \Delta t) - a_{11}(t)}{\Delta t} & \frac{a_{12}(t + \Delta t) - a_{12}(t)}{\Delta t} \\ \frac{a_{21}(t + \Delta t) - a_{21}(t)}{\Delta t} & \frac{a_{22}(t + \Delta t) - a_{22}(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a'_{11}(t) & a'_{12}(t) \\ a'_{21}(t) & a'_{22}(t) \end{bmatrix}. \end{aligned}$$

5) The constant rule (1), sum rule (2), and constant multiple rule (3) also hold for matrix derivatives.

we'll discuss this more carefully on Monday...

Universal product rule: Shortcut to take the derivatives of

- $f(t)\mathbf{x}(t)$ (scalar function times vector function),
- $f(t)A(t)$ (scalar function times matrix function),
- $A(t)\mathbf{x}(t)$ (matrix function times vector function),
- $\mathbf{x}(t) \cdot \mathbf{y}(t)$ (vector function dot product with vector function),
- $\mathbf{x}(t) \times \mathbf{y}(t)$ (cross product of two vector functions),
- $A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of t , and let $*$ be a product operation as above. Then

$$\frac{d}{dt} (A(t) * B(t)) = A'(t) * B(t) + A(t) * B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. Here is one explanation that proves all of those product rules at once:

$$\begin{aligned} \frac{d}{dt} (A(t) * B(t)) &:= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (A(t + \Delta t) * B(t + \Delta t) - A(t) * B(t)) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (A(t + \Delta t) * B(t + \Delta t) - A(t + \Delta t) * B(t) + A(t + \Delta t) * B(t) - A(t) * B(t)) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (A(t + \Delta t) * B(t + \Delta t) - A(t + \Delta t) * B(t)) + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (A(t + \Delta t) * B(t) - A(t) * B(t)) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(A(t + \Delta t) * (B(t + \Delta t) - B(t)) \right) + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (A(t + \Delta t) - A(t)) * B(t) \\ &= \lim_{\Delta t \rightarrow 0} \left(A(t + \Delta t) * \left(\frac{1}{\Delta t} (B(t + \Delta t) - B(t)) \right) \right) + \lim_{\Delta t \rightarrow 0} \left(\frac{1}{\Delta t} (A(t + \Delta t) - A(t)) \right) * B(t) \\ &= A(t) * B'(t) + A'(t) * B(t). \end{aligned}$$

