Math 2280-002
Week 8: 3.6 -3.7, 4.1-4.2
Mon Feb 25
3.6: Forced oscillations

Announcements:

Warm-up Exercise:

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:
We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$ :

  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\, x = 0$$

  which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha) .$$

So for the non-homongeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \quad \Rightarrow x_P = A \cos(\omega\,t)$ because only even derivatives, we don't need

sin $(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A \cos(\omega\,t) + C_0 \cos(\omega_0 t - \alpha_0) .$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0, A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 \left(A \cos(\omega_0\,t) + B \sin(\omega_0\,t)\right)$$
$$\Rightarrow x = x_P + x_H = C\,t \cos(\omega\,t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0) .$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A \cos(\omega\,t) + B \sin(\omega\,t) = C \cos(\omega\,t - \alpha)$ (because the

roots of the characteristic polynomial are never purely imagninary $\pm i\,\omega$ when $c > 0$ ).

- underdamped: $x = x_P + x_H = C \cos(\omega\,t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)$ .
- critically-damped: $x = x_P + x_H = C \cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)$ .
- over-damped: $x = x_P + x_H = C \cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ .

• in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

• if $c$ is small enough and $\omega \approx \omega_0$ then the amplitude $C$ of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

We worked this Exercise on Friday...

<u>Forced undamped oscillations</u>: (We'll discuss forced damped oscillations on Monday next week.)
<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?
1c) Compare your solution and reasoning with the display at the bottom of this page.

soltn from warmup

$x(t) = -5\cos 5t + 5\cos 3t$

$\cos \omega t$
$\omega$ = angular freq $\frac{\text{rad}}{\text{time}}$
$f = \frac{\omega}{2\pi}$ cycles/time
$T = \frac{2\pi}{\omega}$ time/cycle

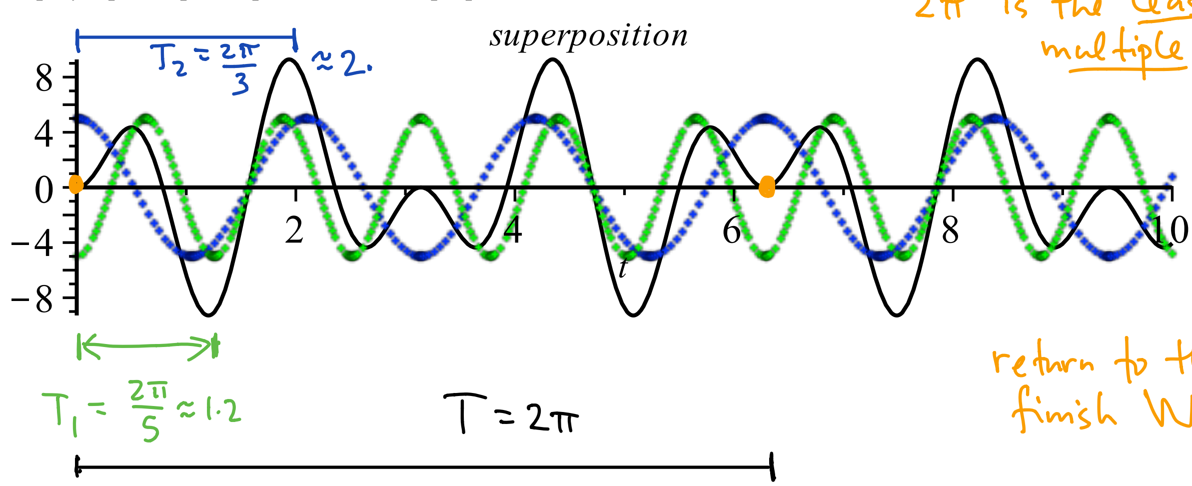
period $T_1 = \frac{2\pi}{5}$

$\cos 5\left(\frac{2\pi}{5}\right) = \cos 2\pi = \cos 0$

$T_2 = \frac{2\pi}{3}$

the period of the sum is (well it looks to be) $2\pi$
$2\pi$ is the <u>least common multiple</u> of $T_1$ & $T_2$
$2\pi = 5T_1$
$2\pi = 3T_2$

```
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```




return to this after finish Wed notes

In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ :

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

Solution:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos(\omega_0 t) - \cos(\omega t)\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}\cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))\\ &= 2\sin(\alpha)\sin(\beta)\ .\end{aligned}$$

Set $\alpha = \frac{1}{2}\left(\omega + \omega_0\right)t$, $\beta = \frac{1}{2}\left(\omega - \omega_0\right)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}\left(\omega - \omega_0\right), \qquad \text{period: } \frac{4\pi}{\left|\omega - \omega_0\right|}\ .$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m\left|\omega^2-\omega_0^2\right|}.$$

$$x(t)=\frac{F_0}{m\left(\omega^2-\omega_0^2\right)}\left(\cos(\omega_0 t)-\cos(\omega t)\right)+x_0\cos(\omega_0 t)+\frac{v_0}{\omega_0}\sin(\omega_0 t)$$

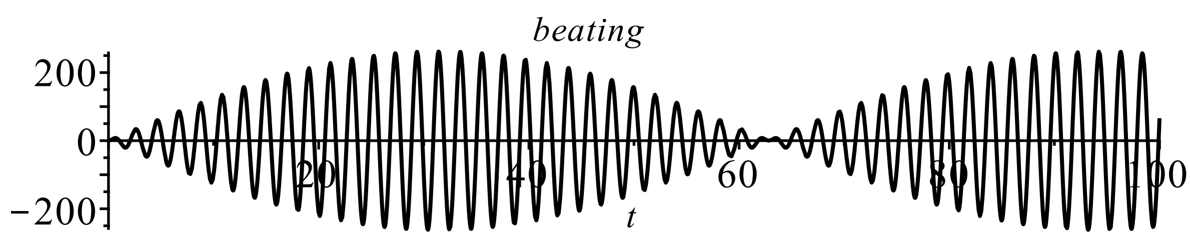
$$x(t)=\frac{F_0}{m\left(\omega^2-\omega_0^2\right)}2\sin\left(\frac{1}{2}(\omega+\omega_0)t\right)\sin\left(\frac{1}{2}(\omega-\omega_0)t\right)+x_0\cos(\omega_0 t)+\frac{v_0}{\omega_0}\sin(\omega_0 t).$$

Exercise 2a) Use one of the formulas above to write down the IVP solution $x(t)$ to

$$x''+9x=80\cos(3.1\,t)$$
$$x(0)=0$$
$$x'(0)=0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

```
> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
```



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using § 5.5, guess

$$+\omega_0^2 \left( \quad x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad \right)$$
$$0 \left( \quad x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad \right)$$
$$+1 \left( \quad x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad \right)$$

$$L(x_p) = t(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right).$$

If we let $\omega \rightarrow \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

Resonance summary:

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m}\cos\left(\omega_0 t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

has solution

$$x(t) = \frac{F_0}{2\,m}\, t \sin\left(\omega_0\, t\right) + x_0 \cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

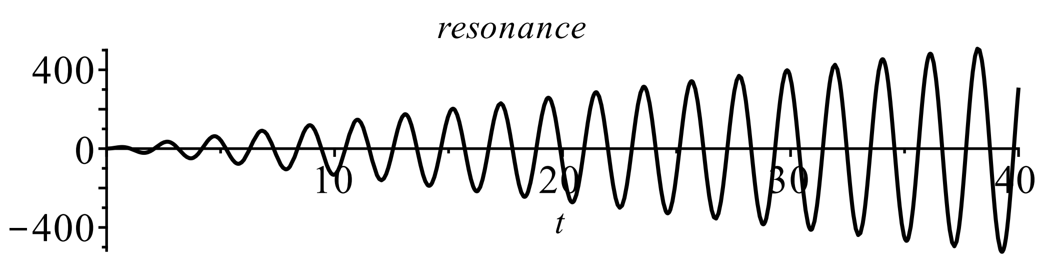
<u>Exercise 3a)</u> Solve the IVP

$$x'' + 9\,x = 80\cos(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Just use the general solution formula above this exercise and substitute in the appropriate values for the various terms.

<u>3b)</u> Compare the solution graph below with the beating graph in <u>exercise 2</u>.

> $plot\left(\frac{40}{3} t\cdot \sin(3\cdot t),\ t=0..40,\ color=black,\ title=`resonance`\right);$




>

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Tues Feb 26
3.6-3.7: Forced oscillations: practical resonance; 3.7 Electrical circuits analog

Announcements:

Warm-up Exercise:

Damped forced oscillations ($c > 0$) for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$
\begin{aligned}
k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\
+\,c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\
+\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] .
\end{aligned}
$$

$$
\begin{aligned}
L(x_P) &= \cos(\omega t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\
&\quad + \sin(\omega t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right) .
\end{aligned}
$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$ .
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}(c_1 t + c_2)$ .
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ .

Important to note:

- The amplitude $C$ in $x_{sp}$ can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle $\alpha$ is always in the first or second quadrant.

From previous page:

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

Since

$$k - m\omega^2 = m\left(\omega_0^2 - \omega^2\right)$$

we may rewrite the steady periodic trig data as

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{F_0}{m}\,\frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{c\,\omega}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

$$k - m\,\omega^2 = m\left(\omega_0^2 - \omega^2\right).$$

Exercise 1) ( a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient $c$! Namely, for $c$ small, when $\omega^2 << \omega_0^2$ we have $\cos(\alpha) \approx 1$, i.e. $\alpha \approx 0$ (in phase with the forcing function) for $x_{sp}$; when $\omega^2 >> \omega_0^2$ we have $\cos(\alpha) \approx -1$, i.e. $\alpha$ near $\pi$ (out of phase with the forcing function); for $\omega \approx \omega_0$, $\sin(\alpha) \approx 1$, i.e. $\alpha \approx \frac{\pi}{2}$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 2) Find $x_{sp}(t)$ for the forced oscillation problem

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0\,.$$

Practical resonance: The steady periodic amplitude $C$ for damped forced oscillations is

$$C(\omega) = C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{F_0}{m}\,\frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

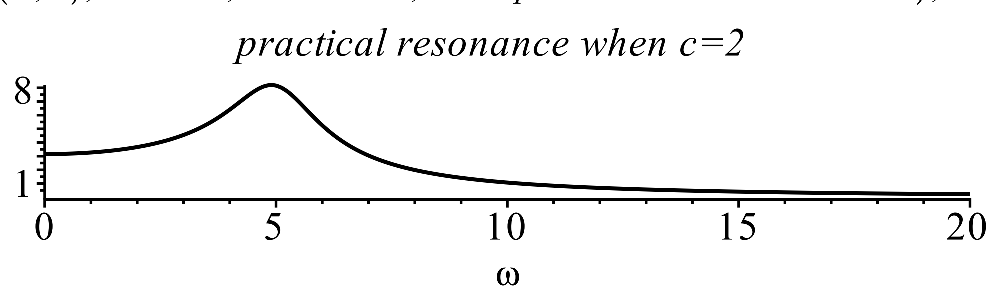
Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k} = \frac{F_0}{m\,\omega_0^2}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$. (Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in the variable $\omega^2$ it will have at most one minimum in the variable $\omega^2$, so $C(\omega)$ will have at most one maximum for non-negative $\omega$. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 3a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient $c$:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

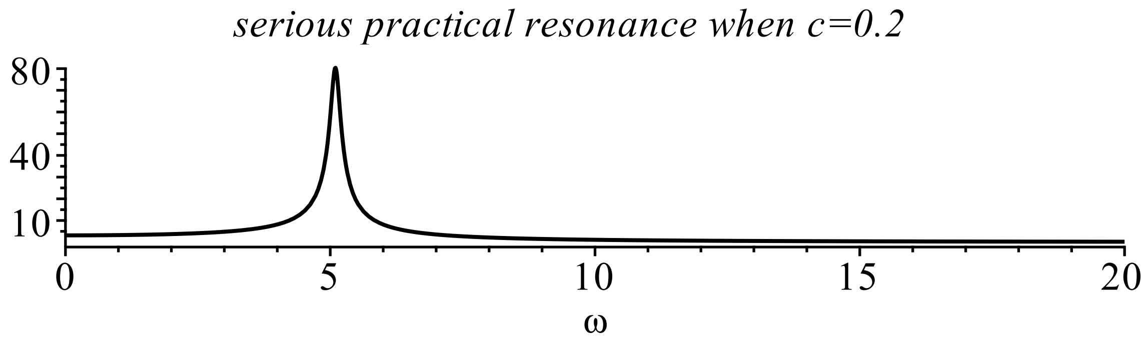
3b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

```
> restart :
> with(plots) :
> C := (ω, c) → 82 / sqrt((26 − ω^2)^2 + c^2·ω^2) :
> plot(C(ω, 2), ω = 0 ..20, color = black, title = `practical resonance when c=2`);
```

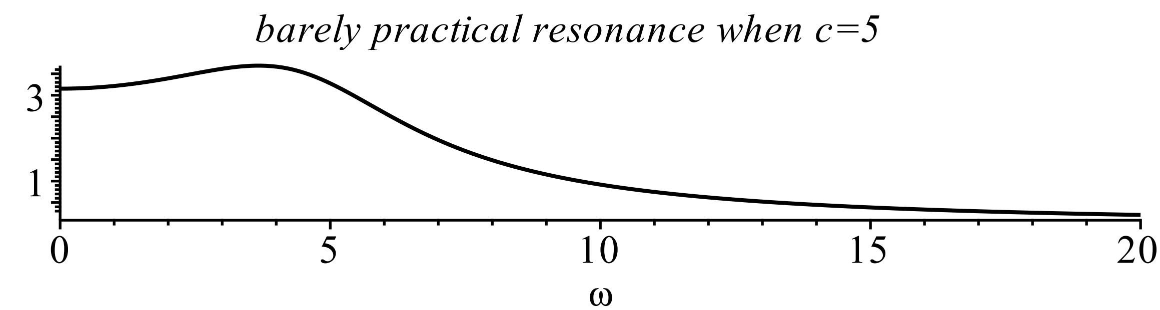



```
>
```

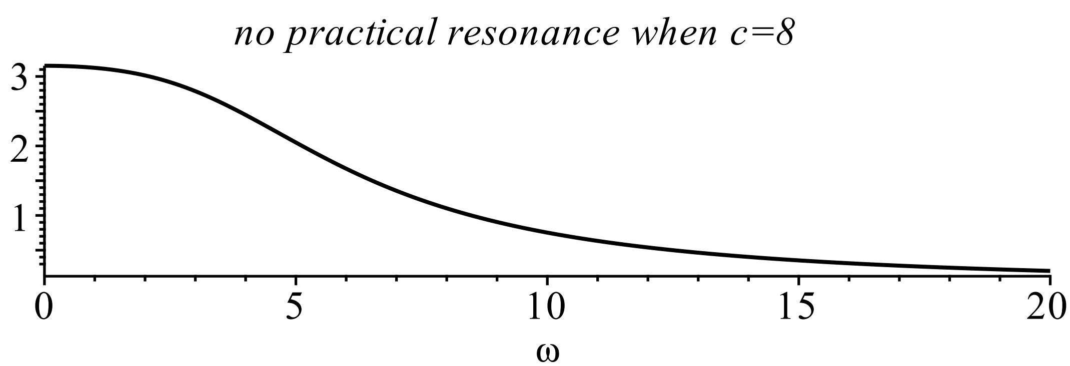
> plot(C(ω, .2), ω = 0 ..20, color = black, title = `serious practical resonance when c=0.2`);



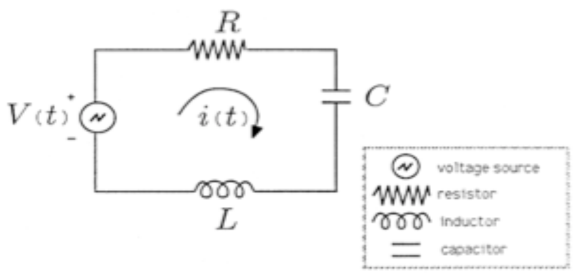

> plot(C(ω, 5), ω = 0 ..20, color = black, title = `barely practical resonance when c=5`);




> plot(C(ω, 8), ω = 0 ..20, color = black, title = `no practical resonance when c=8`);



Section 3.7 <u>The mechanical-electrical analogy:</u> Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....the classical RLC circuit with applied voltage is described in this schematic:




| circuit element | voltage drop | units |
|---|---|---|
| inductor | $L\, I'(t)$ | $L$ Henries ($H$) |
| resistor | $R\, I(t)$ | $R$ Ohms ($\Omega$) |
| capacitor | $\frac{1}{C} Q(t)$ | $C$ Farads ($F$) |

http://cnx.org/content/m21475/latest/pic012.png

<u>Kirchoff's Law</u>: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $$L\, Q''(t) + R\, Q'(t) + \frac{1}{C} Q(t) = V(t) = E_0 \sin(\omega t)$$

For $I(t)$ : $$L\, I''(t) + R\, I'(t) + \frac{1}{C} I(t) = V'(t) = E_0\, \omega \cos(\omega t)\, .$$

We can transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t)\, .$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma)\, , \quad \gamma = \alpha - \frac{\pi}{2}\, .$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}\, .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $\mathrm{I}_0(\omega)$ is called the <u>impedance</u> $Z(\omega)$ of the circuit (because the larger the *impedance*, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$ , i.e.

$$C = \frac{1}{L\,\omega^2}$$ if $L$ is fixed and $C$ is adjustable (old analog radios).

$$L = \frac{1}{C\,\omega^2}$$ if $C$ is fixed and $L$ is adjustable

Both $L$ and $C$ are adjusted in this M.I.T. lab demonstration:

http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

Wed Feb 27
4.1 Introduction to systems of differential equations

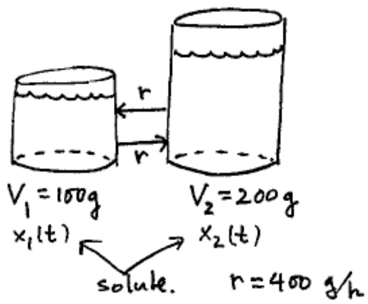
Announcements:

Warm-up Exercise:

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model
and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:




Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$ , write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The example on page 1 is a special case of the general <u>initial value problem for a first order system of differential equations</u>:

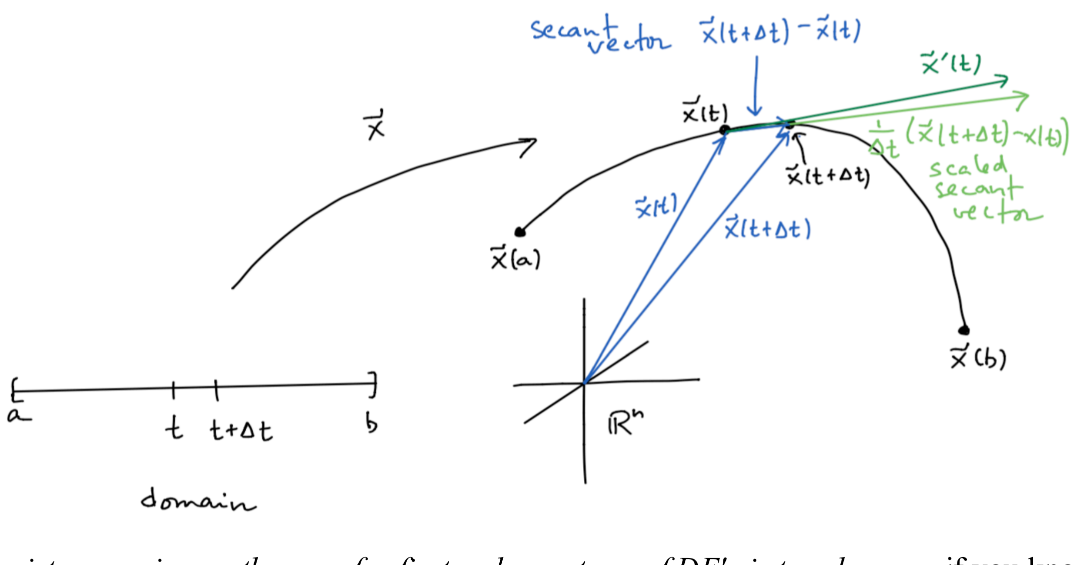
$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

<u>Existence and Uniqueness for solutions to IVP's</u>: The IVP above is a vectorized version of the scalar first order DE IVP that we considered in Chapter 1. In Chapter 1 we understood why (with the right conditions on the right hand side), these IVP's have unique solutions. There is an analogous existence-uniqueness theorem for the vectorized version we study in Chapters 4-6, and it's believable for the same reasons the Chapter 1 theorem seemed reasonable. We just have to remember the geometric meaning of the *tangent* vector $\underline{\boldsymbol{x}}'(t)$ to a parametric curve in $\mathbb{R}^n$ (which is also called the *velocity* vector in physics, when you study particle motion):

*Algebra*:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}\right) = \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

*Geometric interpretation in terms of displacement vectors along a parametric curve:*




*So the existence-uniquess theorem for first order systems of DE's is true because* if you know where you start at time $t_0$, namely $\underline{\boldsymbol{x}}_0$; and if you know your tangent vector $\underline{\boldsymbol{x}}'(t)$ at every later time -in terms of your location $\underline{\boldsymbol{x}}(t)$ and what time $t$ it is, as specified by the vector function $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$; then there should only be one way the parametric curve $\underline{\boldsymbol{x}}(t)$ can develop. This is analogous to our reasoning in Chapter 1 that there should only be one way to follow a slope field, given the initial point one starts at.

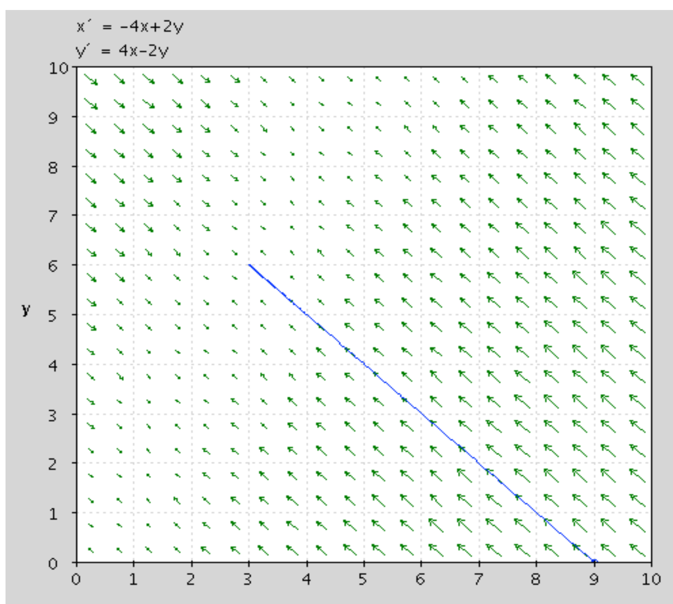
Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4x_1 + 2x_2$$
$$x_2'(t) = 4x_1 - 2x_2$$
$$x_1(0) = 9$$
$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" (tangent vector) vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time *t*. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



Definition: Any first order system of differential equations which can be written in the form

$$\underline{\mathbf{x}}'(t) + P(t)\,\underline{\mathbf{x}} = \underline{\boldsymbol{f}}(t)$$

is called a *first order linear system of DE's*. (Here $\underline{\mathbf{x}}(t)$ and $\underline{\boldsymbol{f}}(t)$ are functions from an interval in $\mathbb{R}$, with range lying in $\mathbb{R}^n$, and $P(t)$ is an $n \times n$ matrix whose entries are functions of $t$. For us $P(t)$ will almost always be a constant matrix. If the system can be written in the form

$$\underline{\mathbf{x}}'(t) + P(t)\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

we say that the linear system of differential equations is *homogeneous*. Otherwise it is *non-homogeneous* or *inhomogeneous*.

Notice that the operator on vector-valued functions $\underline{\mathbf{x}}(t)$ defined by

$$L(\underline{\mathbf{x}}(t)) := \underline{\mathbf{x}}'(t) + P(t)\,\underline{\mathbf{x}}(t)$$

is linear, i.e.

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{y}}(t))$$
$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\mathbf{x}}'(t) + P(t)\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\mathbf{x}}'(t) + P(t)\,\underline{\mathbf{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\mathbf{x}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$$

where $\underline{\mathbf{x}}_P$ is any single particular solution and $\underline{\mathbf{x}}_H$ is the general homogeneous solution.

In the case that $P(t) = -A$ is a constant matrix (i.e. entries don't depend on $t$) , we usually write the homogeneous system as

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}\,.$$

In the case that $A$ is a diagonalizable matrix it turns out we can always find a basis for the homogeneous solution space made of vector-valued functions of the form

$$\underline{\mathbf{x}}(t) = e^{\lambda t}\underline{\mathbf{v}}\,,$$

where $\underline{\mathbf{v}}$ an eigenvector of $A$ and $\lambda$ is its eigenvalue, i.e.

$$A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}.$$

System of DE's:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Candidate solution:

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\,\underline{\boldsymbol{v}}\,,$$

where $\underline{\boldsymbol{v}}$ an eigenvector of $A$ and $\lambda$ is its eigenvalue, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}.$$

We can verify that such an $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\,\underline{\boldsymbol{v}}$ solves the homogeneous DE system above by showing we get a true identity when we substitute it in. We compute the left side of the differential equation:

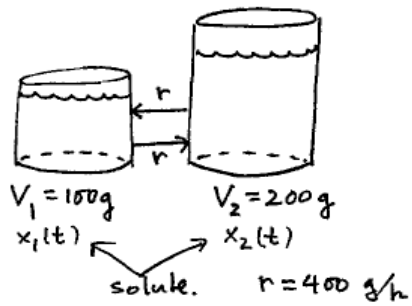
$$\underline{\boldsymbol{x}}(t) = e^{\lambda t} v \quad \Rightarrow \underline{\boldsymbol{x}}'(t) = \lambda\, e^{\lambda t}\,\underline{\boldsymbol{v}}\,.$$

And we compute the right side

$$A\,\underline{\boldsymbol{x}}(t) = A\, e^{\lambda t}\,\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}} = e^{\lambda t}\lambda\,\underline{\boldsymbol{v}}\,.$$

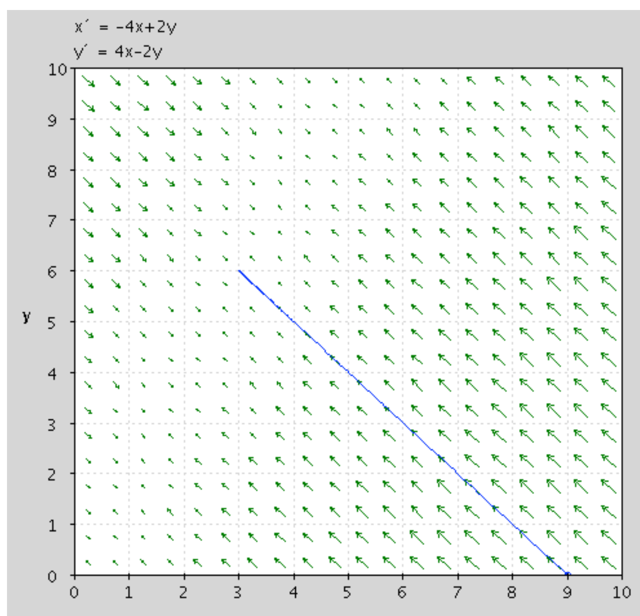
Same!

Exercise 3) Use the eigendata of the matrix in our running example solve the initial value problem of Exercise 2!! Compare your solution $\underline{\mathbf{x}}(t)$ to the parametric curve drawn by pplane.




$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



Fri Mar 1

5.1-5.2 Systems of differential equations and the vector Calculus we need to study them (5.1). Every differential equation or system of differential equations can be converted into a first order system of differential equations (4.1).

Announcements:

Warm-up Exercise:

On Wednesday, we began and maybe finished solving the two-tank IVP example analytically, using eigenvalues and eigenvectors from the matrix $A$ in that problem (!). That is typical of what we will do in section 5.2, to solve the first order system of DE's

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \,.$$

At this point though, it's a good idea to review and extend some differentiation rules you probably learned in multivariable Calculus, when you studied the calculus of parametric curves. This is related to material in section 5.1 of the text, e.g. page 271. Most of the rest of section 5.1 is material you learned in Math 2270 - you may wish to scan it to make sure it's still familiar.

1) If $\underline{\mathbf{x}}(t) = \underline{\boldsymbol{b}}$ is a constant vector, then $\underline{\mathbf{x}}'(t) = \underline{\mathbf{0}}$ for all $t$, and vise-verse. (Because all of the entries in the vector $\underline{\boldsymbol{b}}$ are constants, and their derivatives are zero. And if the derivatives of all entries of a vector are identically zero, then the entries are constants.)

2) Sum rule for differentiation:

$$\frac{d}{dt}\left(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)\right) = \underline{\mathbf{x}}'(t) + \underline{\mathbf{y}}'(t): \qquad \text{Both sides simplify to} \begin{bmatrix} x_1'(t) + y_1'(t) \\ x_2'(t) + y_2'(t) \\ \vdots \\ x_n'(t) + y_n'(t) \end{bmatrix}$$

3) Constant multiple rule for differentiation:

$$\frac{d}{dt}\left(c\,\underline{\mathbf{x}}(t)\right) = c\,\underline{\mathbf{x}}'(t): \qquad \text{Both sides simplify to} \quad c\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

4) Matrix-valued functions sometimes show up and sometimes need to be differentiated. This is done with the limit definition, and amounts to differentiating each entry of the matrix. For example, if $A(t)$ is a $2 \times 2$ matrix, then

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} a_{11}(t+\Delta t) & a_{12}(t+\Delta t) \\ a_{21}(t+\Delta t) & a_{22}(t+\Delta t) \end{bmatrix} - \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\begin{bmatrix} a_{11}(t+\Delta t)-a_{11}(t) & a_{12}(t+\Delta t)-a_{12}(t) \\ a_{21}(t+\Delta t)-a_{21}(t) & a_{22}(t+\Delta t)-a_{22}(t) \end{bmatrix}$$

$$= \lim_{\Delta t \to 0}\begin{bmatrix} \dfrac{a_{11}(t+\Delta t)-a_{11}(t)}{\Delta t} & \dfrac{a_{12}(t+\Delta t)-a_{12}(t)}{\Delta t} \\ \dfrac{a_{21}(t+\Delta t)-a_{21}(t)}{\Delta t} & \dfrac{a_{22}(t+\Delta t)-a_{22}(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}.$$

5) The constant rule (1), sum rule (2), and constant multiple rule (3) also hold for matrix derivatives.

Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\mathbf{x}}(t)$ (scalar function times vector function),
$f(t)A(t)$ (scalar function times matrix function),
$A(t)\underline{\mathbf{x}}(t)$ (matrix function times vector function),
$\underline{\mathbf{x}}(t) \bullet \underline{\mathbf{y}}(t)$ (vector function dot product with vector function),
$\underline{\mathbf{x}}(t) \times \underline{\mathbf{y}}(t)$ (cross product of two vector functions),
$A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of $t$, and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t)*B(t)) = A'(t)*B(t) + A(t)*B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. Here is one explanation that proves all of those product rules at once:

$$\frac{d}{dt}(A(t)*B(t)) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t) + A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*\left(B(t+\Delta t) - B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)*B(t)\right.$$

$$= \lim_{\Delta t \to 0} \left(A(t+\Delta t)*\left(\frac{1}{\Delta t}\left(B(t+\Delta t) - B(t)\right)\right)\right) + \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)\right)*B(t)$$

$$= A(t)*B'(t) + A'(t)*B(t).$$

As an application of what you learned in Math 2270 and of the various vector-matrix differentiation rules on the previous pages we can prove:

Theorem (Section 5.2) Consider the homogeneous system of $n$ differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

where $A$ is an $n \times n$ diagonalizable matrix. Then there is a basis for the solution space to the system of differential equations of the form

$$\left\{ e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1,\ e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2,\ \ldots.\ e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n \right\}.$$

where each vector $\underline{\boldsymbol{v}}_k$ is an eigenvector of $A$ with eigenvalue $\lambda_k$, i.e.

$$A\,\underline{\boldsymbol{v}}_k = \lambda_k\,\underline{\boldsymbol{v}}_k\,.$$

(In the case of complex eigendata, each pair of complex conjugate solutions may be replaced with two real-valued vector valued functions instead.)

proof 1: We checked already that if

$$A\,\underline{\boldsymbol{v}}_k = \lambda_k\,\underline{\boldsymbol{v}}_k$$

then

$$\underline{\boldsymbol{x}}_k(t) := e^{\lambda_k t}\,\underline{\boldsymbol{v}}_k$$

makes the system of DE's

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

a true identity, so each such $\underline{\boldsymbol{x}}_k(t)$ is a solution; since the system is homogeneous all linear combinations of solutions are solutions as well. If we wish to solve the initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

with the linear combination

$$x(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2\, e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \ldots. + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n$$

then at $t = 0$ we are led to the *Wronskian matrix* equation

$$\begin{bmatrix} | & | & \cdots & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \underline{\boldsymbol{x}}_0.$$

Because $\left\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots.\ \underline{\boldsymbol{v}}_n\right\}$ are a basis for $\mathbb{R}^n$, or for $\mathbb{C}^n$ in the case of complex eigendata (that's what it means for a matrix to be *diagonalizable*), this system has unique solutions $\underline{\boldsymbol{c}}$ for all initial vectors $x_0$. Thus all IVPs have unique linear combination solutions, which shows the exponential functions span the solution space and are linearly independent, so a basis for the solution space.

QED

proof 2: This proof ties in to diagonalization concepts from 2270 and finds the general solutions directly instead of seeming to pull them out of the air as we did in the first proof. Begin with the diagonalization identity for the invertible matrix made out of the $\mathbb{R}^n$ or $\mathbb{C}^n$ basis of eigenvectors:

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$ ) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of $A$, then $A$ is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these $\lambda_j$ may be the same, as in the previous example). Let $P$ be the invertible matrix from the previous page, in compressed notation:

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Now let's feed this into our system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

let's change functions in our DE system:

$$\underline{\boldsymbol{x}}(t) = P\,\underline{\boldsymbol{u}}(t), \qquad (\ \underline{\boldsymbol{u}}(t) = P^{-1}\underline{\boldsymbol{x}}(t)\ )$$

Because $P$ is a constant matrix we can use the constant multiple version of the product rule to rewrite the DE system as

$$P\,\underline{\boldsymbol{u}}'(t) = A\,P\,\underline{\boldsymbol{u}}(t).$$

Using $AP = PD$, and then multiplying both sides on the right by $P^{-1}$:

$$P\,\underline{\boldsymbol{u}}'(t) = P\,\mathrm{D}\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{u}}'(t) = \mathrm{D}\,\underline{\boldsymbol{u}}(t).$$

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}.$$

In other words

$$u_1'(t) = \lambda_1 u_1(t),\ u_2'(t) = \lambda_2 u_2(t),\ \ldots\ u_n'(t) = \lambda_n u_n(t).$$

But these are just scalar first order DE's and we know the solutions:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}.$$

So all solutions to the homogeneous first order system are given by

$$\underline{\boldsymbol{x}}(t) = P\,\underline{\boldsymbol{u}}(t)$$

$$\underline{\boldsymbol{x}}(t) = \begin{bmatrix} | & | & \ldots & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \ldots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & \ldots & | \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = c_1 e^{\lambda_1 t} \underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t} \underline{\boldsymbol{v}}_2 + \ldots. + c_n e^{\lambda_n t} \underline{\boldsymbol{v}}_n \quad !!!$$

It is always the case that an initial value problem for single differential equation, or for a system of differential equations is equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example. (See examples and homework problems in section 7.1) This gives us a new perspective on e.g. the way we solved homogeneous differential equations from Chapter 3.

For example, consider this overdamped problem from Chapter 3:

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

Exercise 3a) Solve the IVP above, using Chapter 3 and characteristic polynomial.

3b) Show that if $x(t)$ solves the IVP above, then $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Use your work to write down the solution to the IVP in 3b.

3c) Show that if $[x_1(t), x_2(t)]^T$ solves the IVP in 3b then the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure.

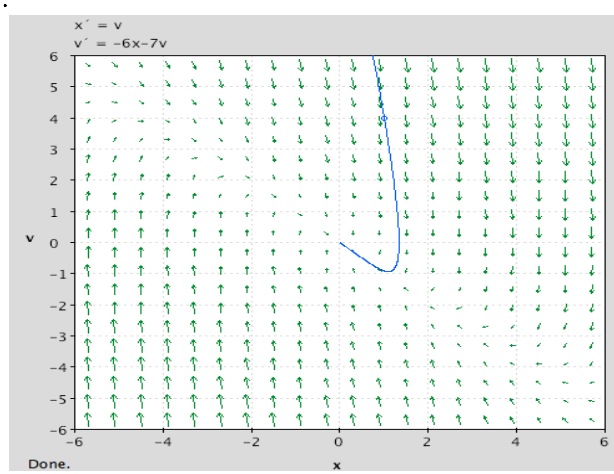
3d) Compare the characteristic polynomial for the homogeneous DE in 3a, to the one for the matrix in the first order system in 3b. It's a mystery (for now)!

$$x''(t) + 7\,x'(t) + 6\,x(t) = 0$$

$$\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}$$

Pictures of the phase portrait for the system in 3b, which is tracking position and velocity of the solution to 3a.

*From pplane, for the system:*




*From Wolfram alpha, for the underdamped second order DE in 3a.*

Input:

$\{x''(t) + 7\,x'(t) + 6\,x(t) = 0,\ x(0) = 1,\ x'(0) = 4\}$

Differential equation solution:

$x(t) = e^{-6t}\,(2\,e^{5t} - 1)$

Plots of the solution: