Math 2280-002
Week 7: 3.5-3.6
Tues Feb 19
3.5: Finding particular solutions to nonhomogeneous constant coefficient linear differential equations.

Announcements:

Warm-up Exercise:

Section 3.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems, and because sometimes a good choice for y_P contains the most essential information in dynamical systems problems).

• The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 3.5.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise as we go through today's notes - *at its core this method is based on the circle of ideas related to the matrix of a linear transformation.*

Exercise 1a) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 3 + 10x .$$

Hint: try $y_P(x) = d_1 + d_2 x$ because L transforms such functions into ones of the same form $b_1 + b_2 x$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 3, b_2 = 10$.

Exercise 1b) Linear algebra interpretation of previous page: Let $\beta = \{y_1(x) = 1, y_2(x) = x\}$ be a basis for the two-dimensional vector space $P_1 = \{y(x) = d_1 + d_2\, x, d_1, d_2 \in \mathbb{R}\}$ of polynomials of degree less than or equal to one. Note that our

$$L(y) := y'' + 4\,y' - 5\,y$$

transforms P_1 back to itself, $L : P_1 \rightarrow P_1$. Use the matrix for L with respect to the basis β to re-find the particular solution $y_P \in P_1$ to

$$L(y) := y'' + 4\,y' - 5\,y = 3 + 10\,x.$$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4\,y' - 5\,y = 10\,x + 3$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x}\,.$$

Hint: try $y_P = d\,e^{2\,x}$ because L transforms functions of that form into ones of the form $b\,e^{2\,x}$, i.e. $L\left(d\,e^{2\,x}\right) = b\,e^{2\,x}$. "$d$" is your "undetermined coefficient" for $b = 14$. (In terms of linear algebra, we are using the fact that for the one-dimensional vector space $V = span\left\{\mathrm{e}^{2\,x}\right\}$, $L(V) = V$.

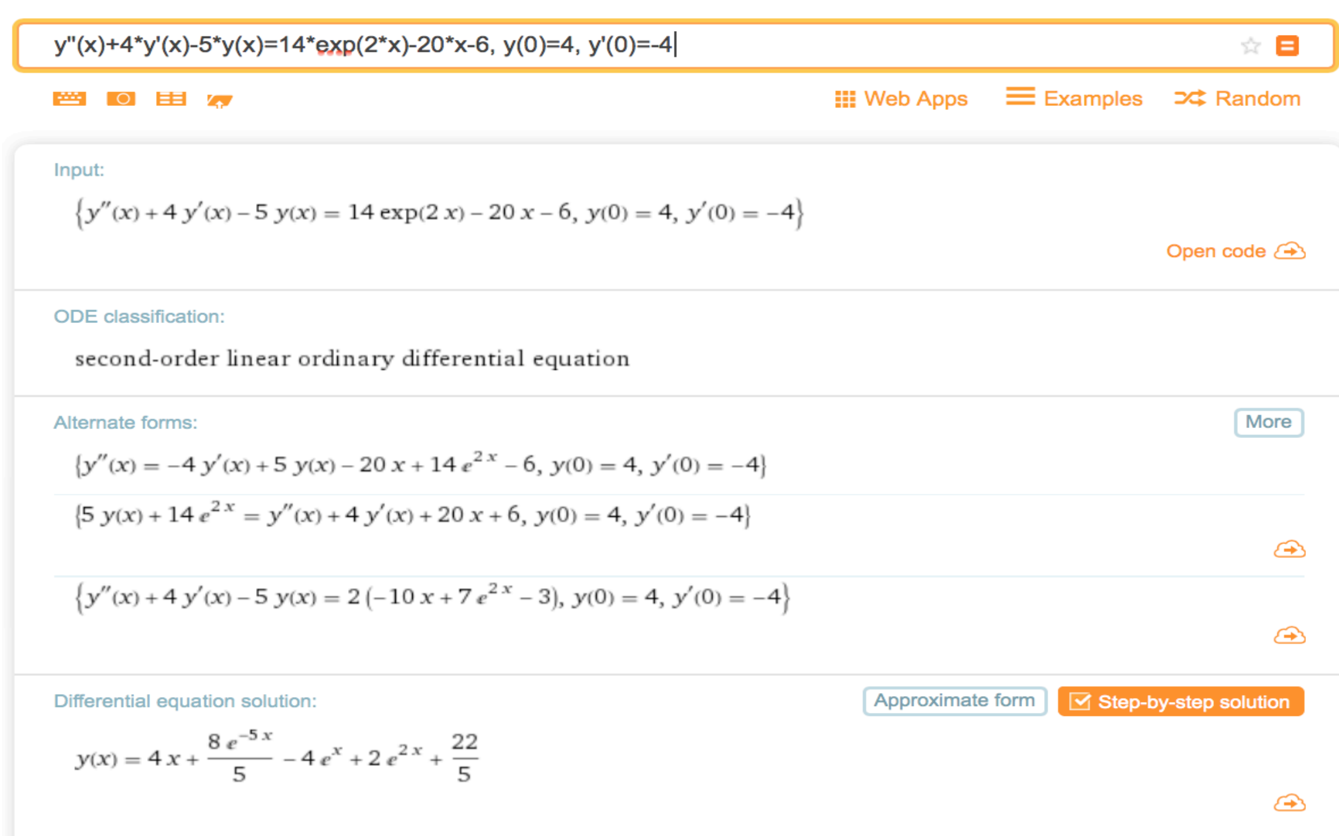
Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6\,.$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6$$
$$y(0) = 4$$
$$y'(0) = -4\,.$$

4c) Check your answer with technology.



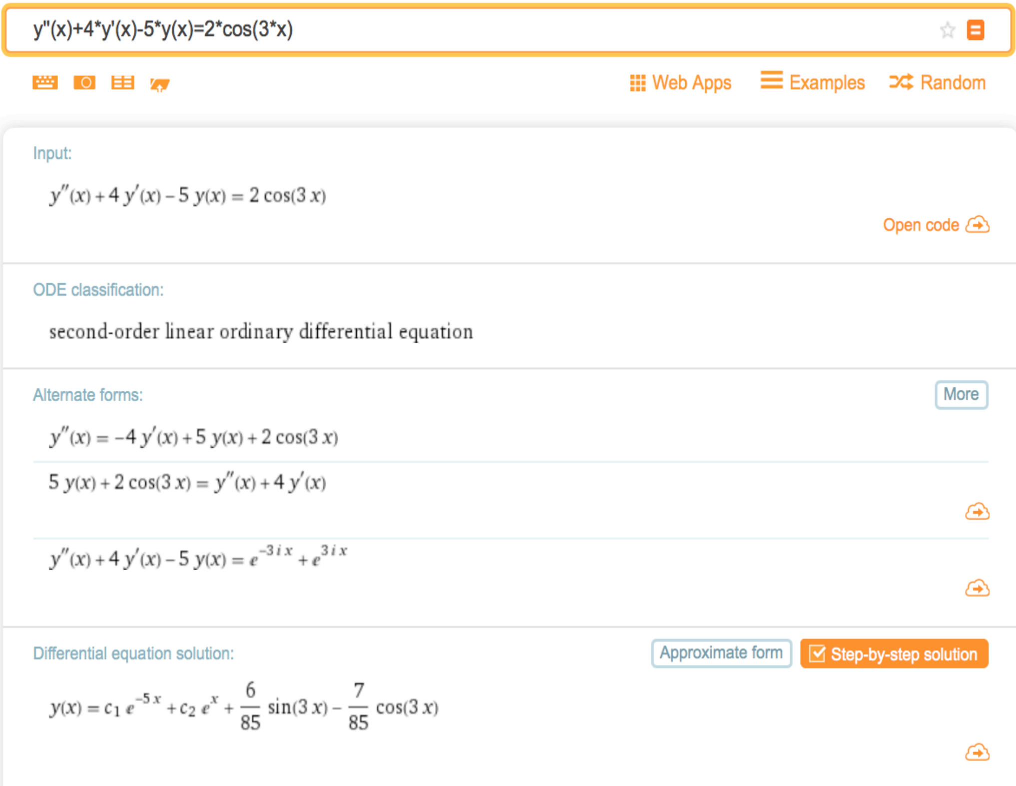
Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \rightarrow V$. If L is an invertible linear transformation then there will be exactly one particular solution y_P in V for $L(y_P) = f$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d e^{2x}$.

- What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?



All of the previous exercises rely on:

<u>Method of undetermined coefficients</u> (<u>base case</u>): Let $L : V \to V$ be a linear transformation, with V a finite dimensional vector space, and let $f \in V$. Then $\exists!\ y_P \in V$ with $L(y_P) = f$ if and only if the only $y \in V$ for which $L(y) = 0$ is $y = 0$.

<u>why:</u> You definitely learned this fact in Math 2270, for the special case of matrix transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\underline{\boldsymbol{x}}) = A_{n\times n}\underline{\boldsymbol{x}}$. (Each non-homogeneous matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution $\underline{\boldsymbol{x}}$ if and only if A reduces to the identity matrix I, if and only if the only solution to the homogeneous equation $\underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$.) The theorem above is a generalization of this fact to general linear transformations $L : V \to V$. In fact, if we pick a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ for V, this base case of undetermined coefficients follows from the fact that the (square) matrix A for L with respect to this basis is invertible if and only if $Nul\ A = \{\underline{\boldsymbol{0}}\}$.

Wed Feb 20:
3.5: Finding y_P for non-homogeneous linear differential equations, continued,

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

Announcements:

Warm-up Exercise:

On Tuesday we discussed the base case of undetermined coefficients:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \rightarrow V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

Exercise 1) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(*assuming* the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

1a) $L(y) = x^3 + 6x - 5$

1b) $L(y) = 4e^{2x} \sin(3x)$

1c) $L(y) = x \cos(2x)$

BUT LOOK OUT

Exercise 2a) Find a particular solution to

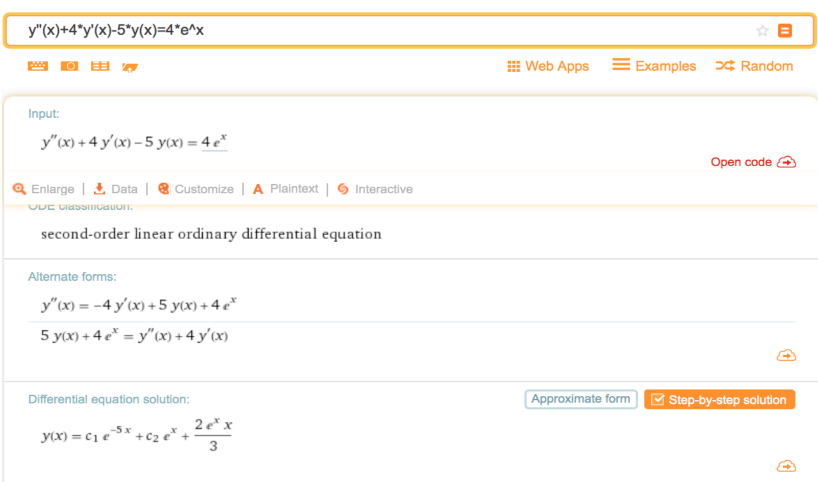
$$y'' + 4\,y' - 5\,y = 4\,e^x .$$

Hint: since $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a\,e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\,x\,e^x$$

and factor $L = D^2 + 4\,D - 5 = [D + 5] \circ [D - 1]$.

2b) check work with technology



Linear algebra to the rescue...just extend our previous discussion:
<u>Theorem</u>: Let

$$L : V \to W$$

be a linear transformation between vector spaces, where $dim\ V = dim\ W = n$. Then for each $f \in W$ there exists unique $y \in V$ solving

$$L(y) = f$$

if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

Proof: Let $\beta = \{y_1, y_2, ..., y_n\}$ be a basis for V. Let $C = \{f_1, f_2, ..., f_n\}$ be a basis for W. Let A be the matrix for L with respect to these two bases. Specfically, A is the matrix which converts β coordinates of input vectors into C coordinates of the outputs of L:

$$[L(y)]_C = A\ [y]_\beta$$

Then the equation $L(y) = f$ has a unique solution if and only if $[L(y)]_C = [f]_C$ does, i.e. if and only if the matrix equation

$$A\ [y]_\beta = [f]_C$$

has a unique solution. And this holds if and only if A reduces to the identity, i.e. if and only if $Nul\ A = \{0\}$, if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

QED

Method of undetermined coefficients ("Rule 2" page 190 text): Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

If L has a factor $(D - r)^s$ and e^{rx} is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 2. (There's also a current homework problem related to this case.) You may also need to use superposition, as in our Tuesday exercises, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = \pm i\omega$
$e^{ax}\left(b_1 \cos(\omega x) + b_2 \sin(\omega x)\right)$	$x^s e^{ax}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = a \pm i\omega$
$b_0 e^{ax}$	$x^s c_0 e^{ax}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right)e^{ax}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)e^{ax}$	$r = a$

Exercise 3) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

3a) $y''' + 2y'' = x^2 + 6x$
(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

Differential equation solution:

$$y(x) = c_3 x^2 + c_2 x + c_1 + \frac{x^5}{180} + \frac{x^4}{12}$$

3b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$
(So the characteristic polynomial for $L(y) = 0$ is
$r^2 - 4r + 13) = (r-2)^2 + 9 = (r-2+3i)(r-2-3i)$.)

Differential equation solution:

$$y(x) = c_1 e^{2x}\sin(3x) + c_2 e^{2x}\cos(3x) + \frac{4}{37}e^{x}\sin(3x) + \frac{24}{37}e^{x}\cos(3x)$$

3c) $y'' + 5y' + 4y = 5\cos(2x) + 4e^x + 5e^{-x}$.
(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$.)

Differential equation solution:

$$y(x) = c_1 e^{-4x} + c_2 e^{-x} + \frac{5 e^{-x} x}{3} + \frac{2 e^x}{5} + \sin(x)\cos(x)$$

Math 2280-002
Fri Feb 22
3.6 forced oscillations

Announcements:

Warm-up Exercise:

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:
We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

  which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\ .$$

So for the non-homogeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A\cos(\omega t)$ because only even derivatives, we don't need

$\sin(\omega t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega t) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0, A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 \left(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\right)$$
$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$ (because the

roots of the characteristic polynomial are never purely imagninary $\pm i\,\omega$ when $c > 0$).

- underdamped: $x = x_P + x_H = C\cos(\omega t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\ .$
- critically-damped: $x = x_P + x_H = C\cos(\omega t - \alpha) + e^{-p\,t}(c_1 t + c_2)\ .$
- over-damped: $x = x_P + x_H = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\ .$

- in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

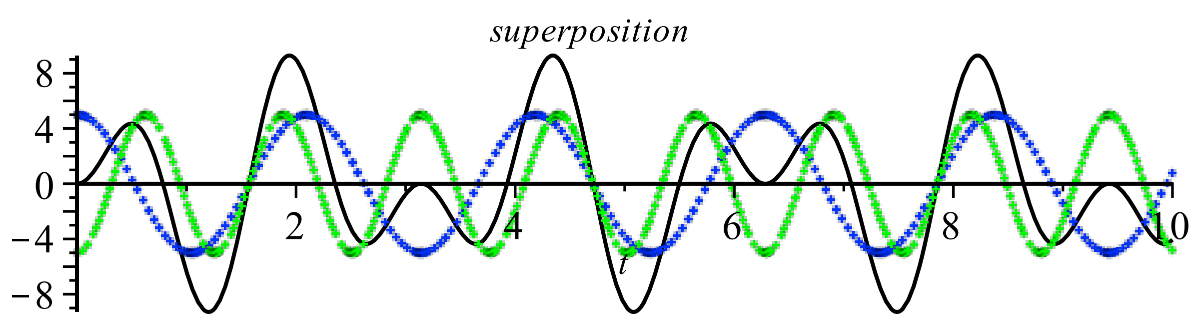
- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

Forced undamped oscillations: (We'll discuss forced damped oscillations on Monday next week.)
Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?
1c) Compare your solution and reasoning with the display at the bottom of this page.

```
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```



In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$:

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

Solution:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega\ t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

There is an interesting <u>beating</u> phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}\cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))\\ &= 2\sin(\alpha)\sin(\beta)\ .\end{aligned}$$

Set $\alpha = \frac{1}{2}\left(\omega + \omega_0\right)t,\ \beta = \frac{1}{2}\left(\omega - \omega_0\right)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}\left(\omega - \omega_0\right), \qquad \text{period: } \frac{4\pi}{\left|\omega - \omega_0\right|}\ .$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m\left|\omega^2-\omega_0^2\right|}.$$

$$x(t) = \frac{F_0}{m\left(\omega^2-\omega_0^2\right)}\left(\cos(\omega_0\, t)-\cos(\omega\, t)\right) + x_0\cos(\omega_0\, t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

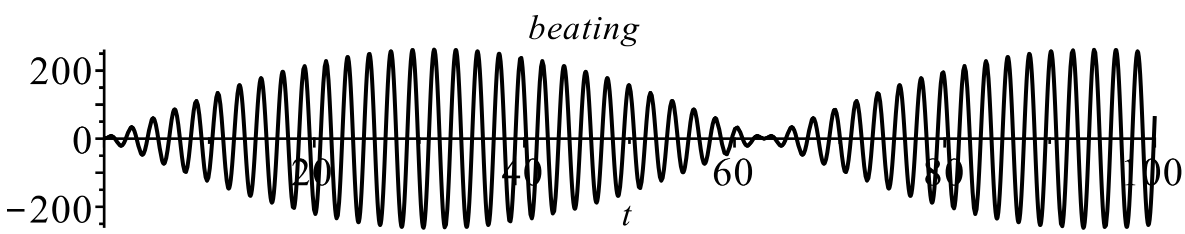
$$x(t) = \frac{F_0}{m\left(\omega^2-\omega_0^2\right)} 2\sin\left(\frac{1}{2}(\omega+\omega_0)t\right)\sin\left(\frac{1}{2}(\omega-\omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

Exercise 2a) Use one of the formulas above to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

```
> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
```



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2\,(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\
0\,(\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\
+1\,(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2 \sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$

$B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

sats $x(0)=0$, $x'(0)=0$, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\dfrac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right).$$

If we let $\omega \rightarrow \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

Resonance summary:

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m}\cos\left(\omega_0 t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

has solution

$$x(t) = \frac{F_0}{2\,m}\, t \sin\left(\omega_0\, t\right) + x_0 \cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

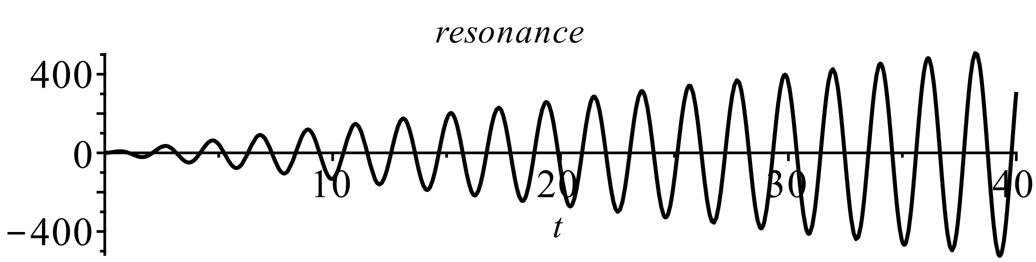
Exercise 3a) Solve the IVP

$$x'' + 9\,x = 80\cos(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Just use the general solution formula above this exercise and substitute in the appropriate values for the various terms.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t\cdot \sin(3\cdot\ t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



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- Next week we will discuss the physics and mathematics of damped forced oscillations

$$m\, x'' + c\, x' + k\, x = F_0 \cos(\omega\, t)\ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)