Math 2280-002
Week 6 Feb 11-13, 3.3-3.4
Exam 1 on Friday Feb 15

Mon Feb 11:
3.3 - 3.4 Continuing discussion of characteristic equation method for homogeneous DE solutions, with focus on complex roots case. Introduction to the unforced mass-spring-damper application, section 3.4

Announcements:

Warm-up Exercise:

3.3 continued and review: How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

Finding a basis for the solution space to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r x}$ yields

$$L(y) = e^{r x}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x} p(r) .$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are topics we finished discussing last week:

Case 1) $p(r)$ has n distinct roots $\{r_1, r_2, \ldots r_n\}$. Then a basis for the solution space to $L(y) = 0$ is given by the corresponding exponential functions:

$$\left\{e^{r_1 x}, e^{r_2 x}, \ldots . e^{r_n x}\right\}.$$

Case 2) $p(r)$ has all real roots, but some are repeated,

$$p(r) = \left(r - r_1\right)^{k_1}\left(r - r_2\right)^{k_2} \cdots \left(r - r_m\right)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get more independent solution functions:

$$e^{r_j x}, x\, e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x} .$$

This yields k_j solutions for each root r_j , so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation, and they are a basis for the solution space.

Case 3) $p(r)$ has complex number roots. This is the hardest, but also most interesting case.The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\, i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we used Taylor-McLauren series reasoning on Friday to justify Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\ .$$

Extending Euler's formula, it makes sense to define

$$e^{a\,+\,b\,i} := e^{a}e^{b\,i} = e^{a}\left(\cos(b) + i\sin(b)\right)$$

for $a, b \in \mathbb{R}$. (If you're a wiz with the binomial theorem, you can also justify that identity by expanding the MacLauren series for $\mathrm{e}^{a\,+\,bi}$.) So for $x \in \mathbb{R}$,

$$e^{(a\,+\,b\,i)x} = e^{a\,x}\left(\cos(b\,x) + i\sin(b\,x)\right) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\ .$$

We compute derivatives for for complex-valued functions $f(x) + i\,g(x)$ with the limit definition,

$$\frac{d}{dx}\left(f(x) + i\,g(x)\right) := \lim_{h\,\to\,0}\frac{\left(f(x+h) + i\,g(x+h)\right) - \left(f(x) + i\,g(x)\right)}{h}$$

and the result after elementary algebra and limit theorems for sums is the computational conclusion

$$\mathrm{D}_x\left(f(x) + i\,g(x)\right)f'(x) + i\,g'(x)\ .$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. This is because all of these rules are consequences of the limit definition of derivative, algebraic manipulations, and limit theorems. The following specific derivative does need to be computed separately for our uses, though:

<u>Exercise 1)</u> Check that $\mathrm{D}_x\left(e^{(a\,+\,b\,i)x}\right) = (a + b\,i)e^{(a\,+\,b\,i)x}$, i.e.

$$\mathrm{D}_x\,e^{r\,x} = r\,e^{r\,x}$$

even if r is complex. (So also $\mathrm{D}_x^2\,e^{r\,x} = \mathrm{D}_x r\,e^{r\,x} = r^2\,e^{r\,x}$, $\mathrm{D}_x^3\,e^{r\,x} = r^3\,e^{r\,x}$, etc.)

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y .$$

Exercise 1 shows that for complex $r = a + b\,i \quad (a, b \in \mathbb{R})$,

$$L\left(e^{rx}\right) = e^{rx}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx}\,p(r)\,,$$

just as was the case for real numbers r. So if $r = a + b\,i$ is a complex root of $p(r)$ then e^{rx} is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we compute derivatives of complex functions, we can compute in this case that

$$0 + 0\,i = L\left(e^{rx}\right) = L\left(e^{ax}\cos(b\,x) + i\,e^{ax}\sin(b\,x)\right)$$

$$= L\left(e^{ax}\cos(b\,x)\right) + i\,L\left(e^{ax}\sin(b\,x)\right).$$

For each $x \in \mathbb{R}$ this identity yields an equality of complex numbers, i.e. separate equalities of their real and imaginary parts. We deduce the identities

$$0 = L\left(e^{ax}\cos(b\,x)\right)$$
$$0 = L\left(e^{ax}\sin(b\,x)\right).$$

Upshot: If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{ax}\cos(b\,x)$$
$$y_2 = e^{ax}\sin(b\,x)$$

are two (independent) solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to $y_1, -y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, ..., a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a\,x}\cos(b\,x),\ e^{a\,x}\sin(b\,x) \\ x\,e^{a\,x}\cos(b\,x),\ x\,e^{a\,x}\sin(b\,x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1}e^{a\,x}\cos(b\,x),\ x^{k-1}e^{a\,x}\sin(b\,x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 2) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 4\,y = 0\,.$$

(You were told a basis in one of your custom homework problems last week....now you know where it came from.)

Exercise 3) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\,.$$

Exercise 4) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

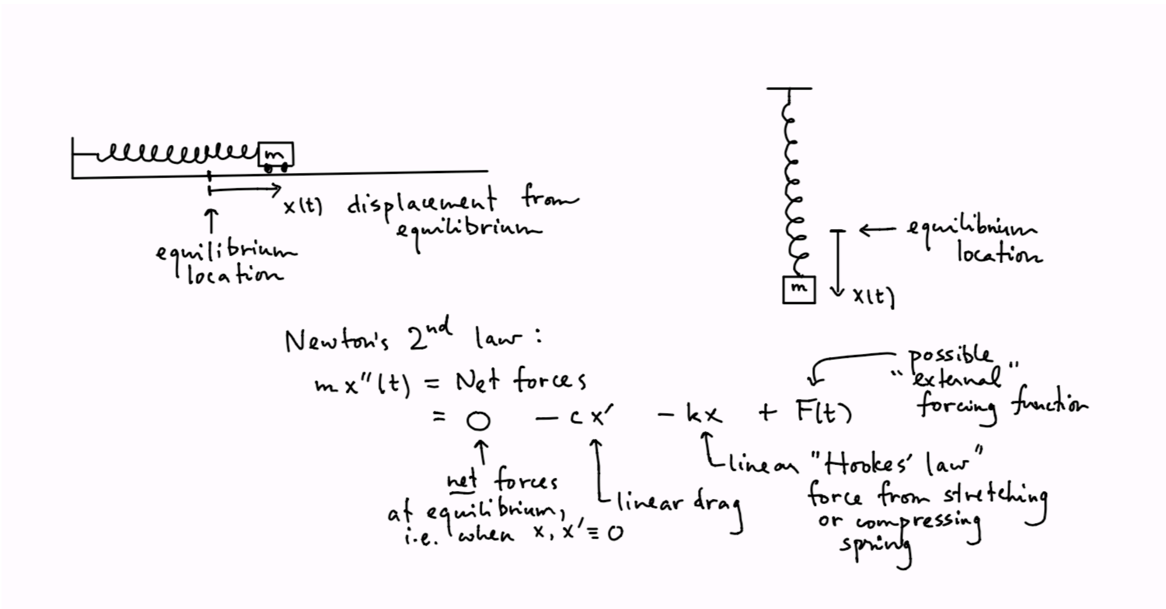
$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r-2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?

3.4: Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$: Notice that we've changed the lettering for our variable and for our functions, as the text does when we hit applications settings. This is the unforced damped mass-spring differential equation and initial value problem:

$$m\,x'' + c\,x' + k\,x = 0$$
$$x(0) = x_0$$
$$x'(0) = v_0$$



In section 3.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0, x' = 0$, for which the net forces must be zero, as we discussed last week. Notice that $c \geq 0, k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t), x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

$$m\,x'' + c\,x' + k\,x = 0\,.$$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ pocesses, because this changes the nature of the basis functions that result for the homogeneous solution space. We'll work a specific sequence of examples first, in which we keep the initial conditions, mass, and spring constant fixed, and vary the damping coefficient c. These examples are part of general considerations which we will follow up on in more detail, tomorrow.

Exercise 4) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

4a) Show that the mass' motion is described by $x(t)$ that solves

$$\begin{aligned} x'' + 9\,x &= 0 \\ x(0) &= 1 \\ x'(0) &= \frac{3}{2} \end{aligned}$$

and solve the IVP. (This is the cae of *simple harmonic motion,* which occurs when the system is *undamped*, $c = 0$, so that the roots of the characteristic polynomial are (pure) imaginary numbers.)

For the following three IVP's write down the general solution to the homogeneous problem and verify that you could solve the IVP's.

4b) (*under-damped:* The damping coefficient c is small enough so that the roots of the characteristic polynomial are complex, with negative real part.)

$$\begin{aligned} x'' + 2\,x' + 9\,x &= 0 \\ x(0) &= 1 \\ x'(0) &= \frac{3}{2}\,. \end{aligned}$$

4c) (*critically-damped:* The damping coefficient is at the special "critical" value for which the characteristic polynomial has a double real root instead of two complex roots - slightly smaller c's, or two real roots - slightly larger c's.)

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

4d) (*over-damped:* The damping coefficient is greater than the critical value, so that the characteristic polynomial has two positive real roots.)

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

Wolfram alpha: It will solve any of the DE IVP's on the previous pages, for example the underdamped one:

x"(t)+2*x'(t)+9*x(t)=0, x(0)=1, x'(0)=3/2

Browse Examples | Surprise Me

Differential equation solution:

$$x(t) = \frac{1}{8} e^{-t} \left(5\sqrt{2}\sin\left(2\sqrt{2}\,t\right) + 8\cos\left(2\sqrt{2}\,t\right)\right)$$

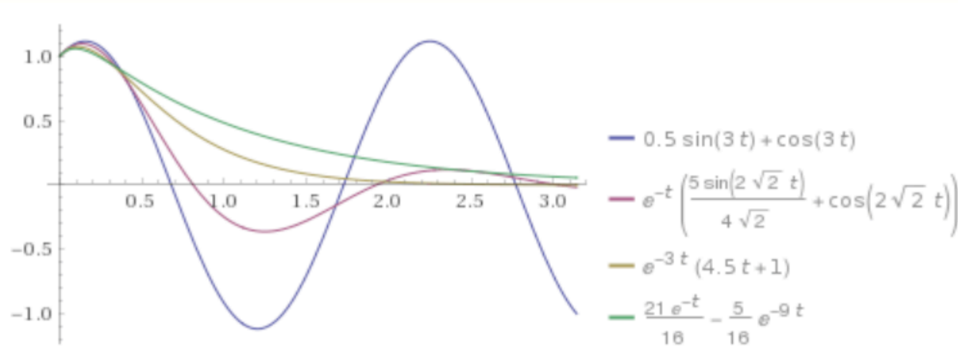
Display of graphs of all 4 solution functions:

Input interpretation:

plot	$\cos(3t) + 0.5\sin(3t)$	$t = 0$ to π
	$\exp(-t)\left(\frac{5}{8}\sqrt{2}\sin\left(2\sqrt{2}\,t\right) + \cos\left(2\sqrt{2}\,t\right)\right)$	
	$\exp(-3t)(1 + 4.5t)$	
	$\frac{21}{16}\exp(-t) - \frac{5}{16}\exp(-9t)$	

Enlarge | Data | Customize | Plaintext | Interactive

Plot:



Tues Feb 12:
3.4 continued....systematic summary of the physical phenomena associated with unforced damped mass-spring configurations.

Announcements:

Warm-up Exercise:

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has purely imaginary roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\, x = 0$$

is

$$x(t) = A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\,.$$

It's worth learning to recognize the undamped DE, and the trigonometric solutions, as it's easy to understand why they are solutions and you can then skip the characteristic polynomial step.

<u>Exercise 1a</u> Write down the general homogeneous solution $x(t)$ to the differential equation

$$x''(t) + 4\,x(t) = 0.$$

<u>1b</u>) What is the general solution to $\theta(t)$ to

$$\theta''(t) + 10\,\theta(t) = 0.$$

The motion exhibited by the solutions

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

to the undamped oscillator DE

$$x''(t) + \omega_0^2 x(t) = 0$$

is called <u>simple harmonic motion.</u> The reason for this name is that $x(t)$ can be rewritten in "amplitude-phase form" as

$$x(t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why this is so, equate the two forms and see how the coefficients A, B, C and phase angle α must be related:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha).$$

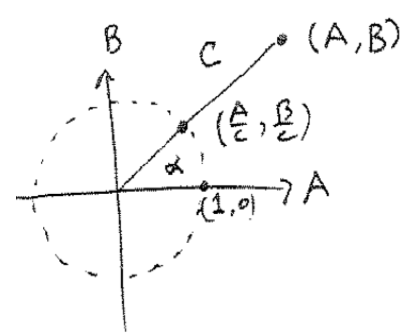
<u>Exercise 2)</u> Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two expressions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha), \quad \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

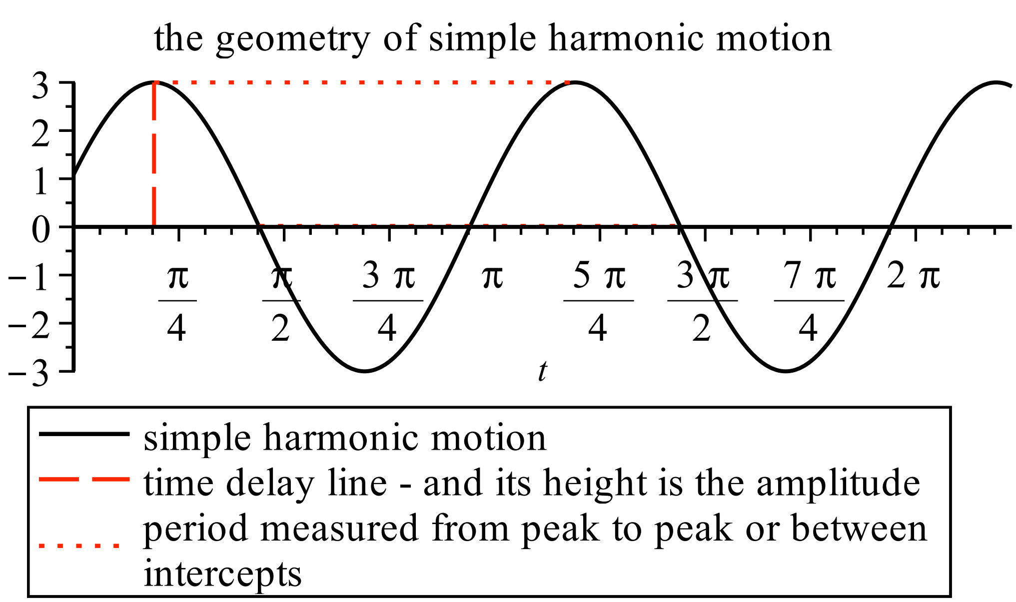
and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The *phase angle* α is the radians of $\omega_0 t$ on the unit circle, so that $\cos(\omega_0 t - \alpha)$ evaluates to 1. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right along the t-axis in order to obtain the graph of $x(t)$. Note that

ω_0 = angular velocity units: radians/time

f = frequency $= \dfrac{\omega_0}{2\pi}$ units: cycles/time

T = period $= \dfrac{2\pi}{\omega_0}$ units: time/cycle.



Exercise 3) Yesterday we solved the differential equation IVP

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

Its solution is

$$x(t) = \cos(3t) + \frac{1}{2}\sin(3t).$$

Convert the formula for $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the Wolfram alpha output on the next page x"(t)+9*x(t)=0, x(0)=1, x'(0)=3/2

Web Apps | Examples | Random

Input:

$\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ x'(0) = \frac{3}{2}\}$

Open code

Autonomous equation:

$x''(t) = -9\,x(t)$

Autonomous equation »

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

$\{x''(t) = -9\,x(t),\ x(0) = 1,\ x'(0) = \frac{3}{2}\}$

$\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ 2\,x'(0) = 3\}$

Differential equation solution:

Step-by-step solution

$x(t) = \frac{1}{2}\sin(3\,t) + \cos(3\,t)$

WolframAlpha computational knowledge engine.

plot cos(3*t)+.5*sin(3*t),t=0..Pi

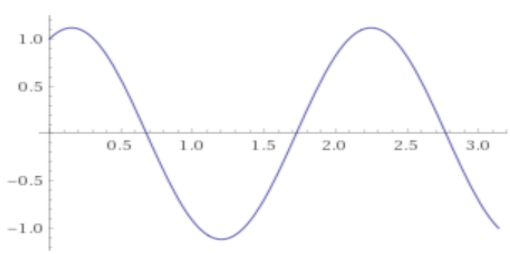
Web Apps | Examples | Random

Input interpretation:

plot	$\cos(3\,t) + 0.5\,\sin(3\,t)$	$t = 0$ to π

Open code

Plot:



Case 2: Unforced mass-spring system with damping: (We did concrete examples of each of the three subcases below yesterday.)

- 3 possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

$$m\,x'' + c\,x' + k\,x = 0$$

$$x'' + \frac{c}{m}\,x' + \frac{k}{m}\,x = 0$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \dfrac{c}{2\,m},\ \omega_0^2 = \dfrac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

Case 2a) $(p^2 < \omega_0^2$, or $c^2 < 4\,m\,k)$ underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$, the undamped angular frequency.

$$x(t) = e^{-p\,t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-p\,t}C\cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}\,.$$

<u>Case 2b)</u> ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) <u>critically damped.</u> Double real root $r_1 = r_2 = -p = -\frac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once. The critically damped case is the transition between underdamped and overdamped:

<u>Case 2c)</u> ($p^2 > \omega_0^2$, or $c^2 > 4\,m\,k$). <u>overdamped.</u> In this case we have two <u>negative real</u> roots

$$r_1 = -p - \sqrt{p^2 - \omega_0^2} < 0$$

$$r_1 < r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$$

and

$$x(t) = c_1 e^{r_1\,t} + c_2\,e^{r_2\,t} = e^{r_2\,t}\left(c_1 e^{\left(r_1 - r_2\right)t} + c_2\right).$$

- solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once, just like in the critically damped case. We did a specific example of all possible cases yesterday, and it may help to review the final picture at the end of Monday's notes.

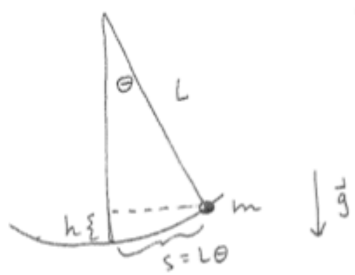
Wed Feb 13:
3.4 experiments, and review for exam

Announcements:

Warm-up Exercise:

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of energy, as indicated below. Conservation of energy is an important tool in deriving differential equations, in a number of different contexts. Today we will test both the pendulum model and the mass-spring model with actual experiments (in the undamped cases), to see if the predicted periods $T = \frac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const}.$

$\frac{1}{2} m v^2 + m g h = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2} m L^2 (\theta'(t))^2 + mgL(1 - \cos(\theta(t))) \equiv \text{const}$

D_t: $m L^2 \theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'(L\theta'' + g\sin\theta) \equiv 0$

($mL\theta' \neq 0$ except at isolated times)

$\leadsto$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize:

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

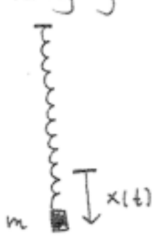
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\sin\theta \approx \theta$ $\quad \theta$ small

is excellent approx

(alternating series test)

2) hanging mass-spring:



$m x'' = -kx$

$m x'' + kx = 0$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

> $L := 1.526;$
> $g := 9.806;$
> $\omega := \sqrt{\frac{g}{L}}$; # *radians per second*
> $f := evalf(\omega/(2\cdot \text{Pi}))$; # *cycles per second*
> $T := 1/f$; # *seconds per cycle*

$$L := 1.526$$
$$g := 9.806$$
$$\omega := 2.534945798$$
$$f := 0.4034491542$$
$$T := 2.478627082 \quad \textbf{(1)}$$

Experiment:

Mass-spring:

compute Hooke's constant:

> $98.7 - 83.4$; #*displacement from extra 50g*

$$15.3 \quad \textbf{(2)}$$

> $k := \frac{.05\cdot 9.806}{.153}$; # *solve* $k\cdot x = m\cdot g$ *for k.*

$$k := 3.204575163 \quad \textbf{(3)}$$

> $m := .1$; # *mass for experiment is 100g*
> $\omega := \sqrt{\frac{k}{m}}$; # *predicted angular frequency*
> $f := evalf\left(\frac{\omega}{2\cdot \text{Pi}}\right)$; # *predicted frequency*
> $T := \frac{1}{f}$; # *predicted period*

$$m := 0.1$$
$$\omega := 5.660896716$$
$$f := 0.9009596945$$
$$T := 1.109927565 \quad \textbf{(4)}$$

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>

Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\,s\,\mathrm{d}s = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\,x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}\left(\mu\,x'(t)\right)^2\left(m_{spring}\,\mathrm{d}\mu\right)$$

$$= \frac{1}{2}m_{spring}(x'(t))^2\int_0^1 \mu^2\,d\,\mu = \frac{1}{6}m_{spring}(x'(t))^2 \quad .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3}m_{spring}\right)(x'(t))^2 + \frac{1}{2}k\,x^2 = \frac{1}{2}M(x'(t))^2 + \frac{1}{2}k\,x^2\,,$$

where

$$M = m + \frac{1}{3}m_{spring}$$

$D_t(TE) = 0 \Rightarrow$

$$M\,x'(t)\,x''(t) + k\,x(t)\,x'(t) = 0\,.$$
$$x'(t)\,(M\,x'' + k\,x) = 0\,.$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M\,x'' + k\,x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3}m_{spring}}}\quad .$$

Does this lead to a better comparison between model and experiment?

```
> ms := .0103;  # spring has mass 10.3 g
  M := m + 1/3·ms;  # "effective mass"
```

$$ms := 0.0103$$
$$M := 0.1034333333 \quad (5)$$

> $\omega := \sqrt{\frac{k}{M}}$; # *predicted angular frequency*
$f := evalf\left(\frac{\omega}{2 \cdot \text{Pi}}\right)$; # *predicted frequency*
$T := \frac{1}{f}$; # *predicted period*

$$\omega := 5.566150833$$
$$f := 0.8858804190$$
$$T := 1.128820525 \quad (6)$$

>

Exam 1 is this Friday February 15, from 12:50-1:50 p.m.

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.4. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems.

I will have posted one or two practice exams and solutions, from recent times I've I taught Math 2280. Theyshould give you a feel for how I structure exams and address course topics. I'll go over a practice exam on Thursday February 14, 1:00-2:20, location TBA.

Exam 1 Review Questions

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order (or higher order) DE?

1b) How do you check whether a function solves a differential equation? An initial value problem?

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What does the existence-uniqueness theorem say? What can cause solutions to not exist beyond a certain input variable value?

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

1f) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

2a) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field? What models did we study, related to population dynamics?

2b) Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?

3a) For functions $y(x)$, why is

$$L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$$

called linear?

3b) For linear operators L, why is the general solution to

$$L(y) = f$$

given by $y = y_p + y_H$, where y_p is any single particular solution, and y_H is the general solution to the homogeneous problem?

3c) For the differential operator L above, what is the dimension of the solution space to the homogeneous DE

$$L(y) = 0?$$

What does this have to do with the existence-uniqueness theorem?

3d) Can you check whether collections of functions are linearly independent?

3e) What's the Wronskian matrix? How does it arise in studying initial value problems?

3f) What's the algorithm for finding the solution space to

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

(when all the a_j are constants)? What is Euler's formula, and what does it have to do with this discussion? How are repeated roots to the characteristic polynomial handled? Why are the solutions that the algorithm creates linearly independent?

3g) For the application to unforced (but possibly damped) mass-spring configurations

$$m\, x''(t) + c\, x'(t) + k\, x(t) = 0$$

what sorts of phenomena arise? Can you convert to amplitude-phase form for simple harmonic motion? Can you describe the important quantities for simple harmonic motion? How are damping phenomena classified? Can you solve IVPs?