Math 2280-001
Week 3: Jan 22-25, sections 2.1-2.3

Tues Jan 22
Finish improved population models 2.1, and discuss input-output applications 1.5.

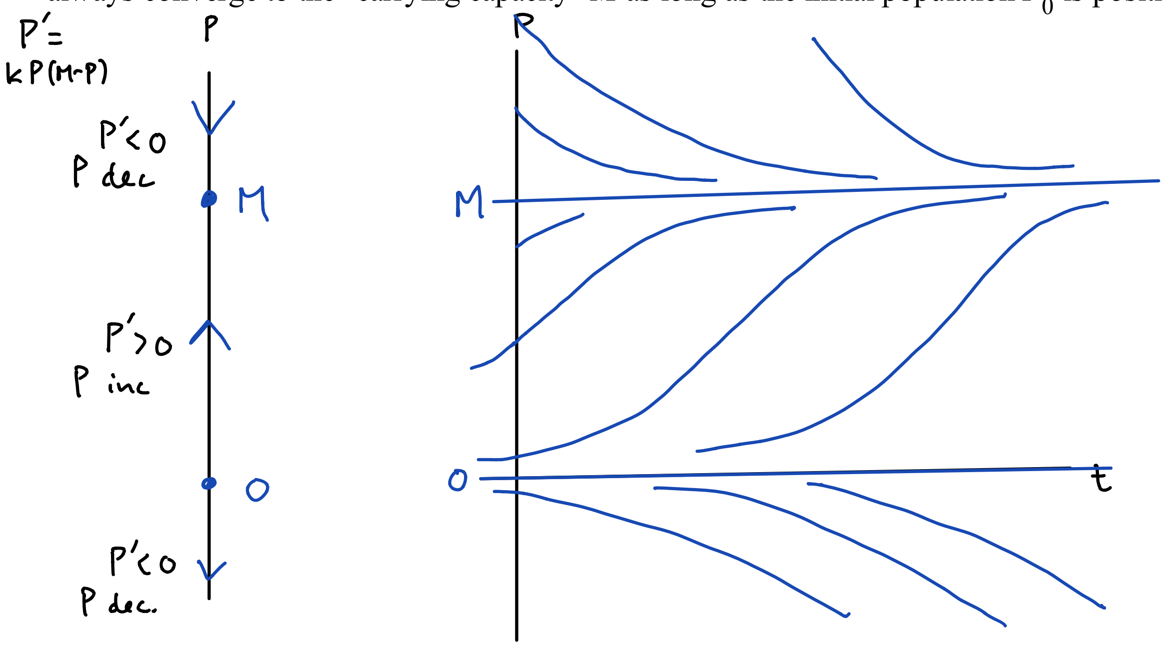
Announcements:

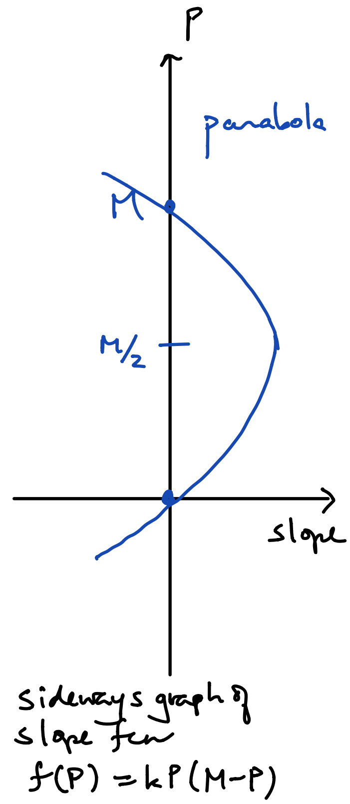
Warm-up Exercise:

Friday recap: We discussed population models that can be more effective than the exponential growth/decay model in certain applications, in particular the logistic growth equation,

$$P'(t) = k\,P\,(M - P)$$

where k, M are positive constants. By analyzing the slope field and a compressed 1-dimensional *phase diagram* related to the slope field, we deduced that it was likely that solutions to the logistic DE IVP's always converge to the "carrying capacity" M as long as the initial population P_0 is positive.





Then we used partial fractions and separation of variables to solve the IVP

$$P'(t) = k\,P\,(M - P)$$
$$P(0) = P_0,$$

finding that the solution

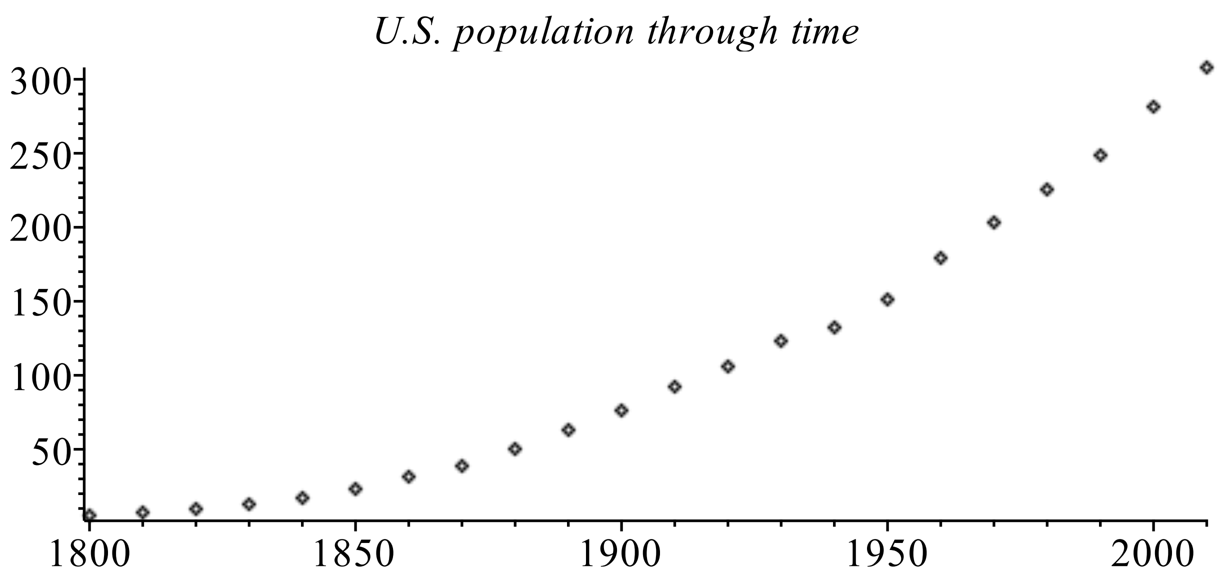
$$P(t) = \frac{M P_0}{\left(M - P_0\right)e^{-Mkt} + P_0}$$

is consistent with our predictions based on the slope field and the phase diagram.

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0 \, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \quad \textbf{(1)}$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \to 5.308\, e^{0.02664\, t} \quad \textbf{(2)}$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \to \frac{M\, P0}{P0 + (M - P0)\, e^{-M k t}} \quad \textbf{(3)}$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \quad \textbf{(4)}$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
     #equation (11) page 84.
```

$$M := 188.12$$

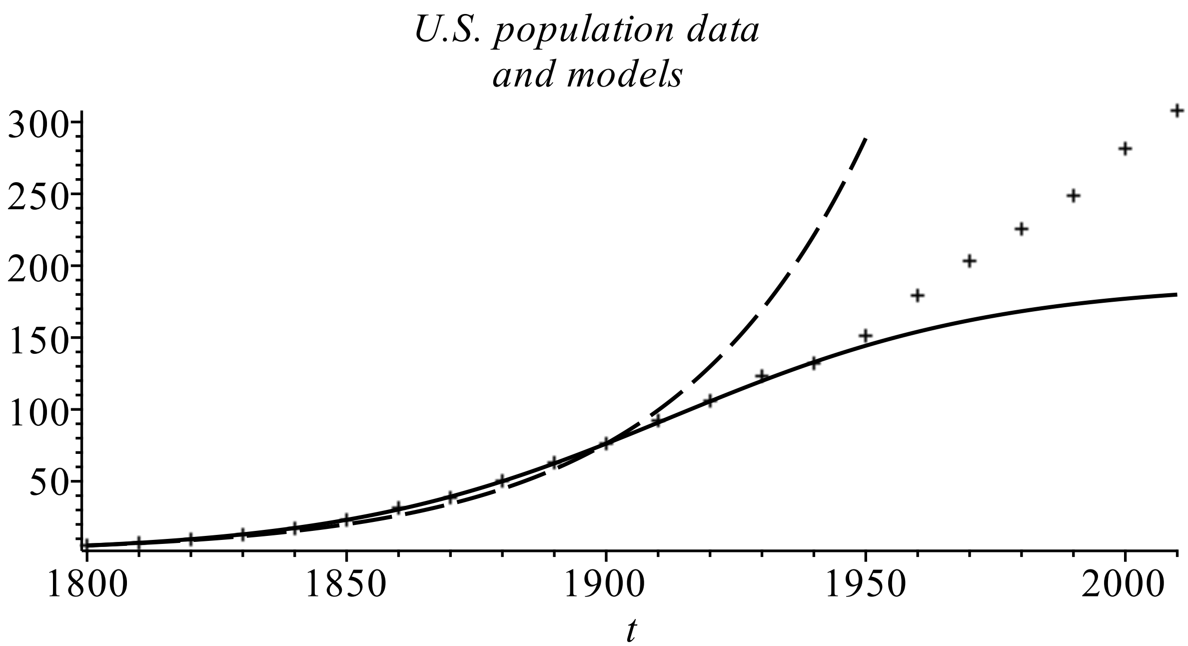
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, e^{-0.031551\, t}} \quad \textbf{(5)}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources.

```
> plot1 := plot(P1(t-1800), t = 1800..1950, color = black, linestyle = 3) :
  #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t-1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
```

Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

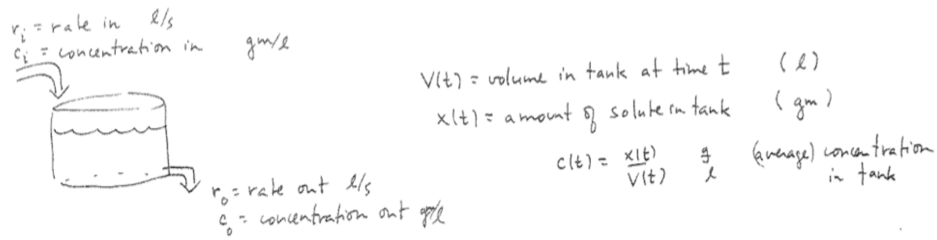
Section 1.5 modeling:

An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_0 . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l} \ .$$

See the diagram below.



<u>Exercise 1:</u> Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants.

<u>Exercise 2</u> (we did this last week as a warm-up exercise). The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t} .$$

Exercise 3 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols
c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \dfrac{km^3}{y}$

b) Solve the IVP, and then answer the question.

Wed Jan 23
2.2 phase portrait analysis and applications

Announcements:

Warm-up Exercise:

2.2: Autonomous Differential Equations.

Recall, that a general first order DE for $x = x(t)$ is written in standard form as

$$x' = f(t, x) \;,$$

which is shorthand for $x'(t) = f(t, x(t))$.

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \;.$$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3\,x - x^2$

1b) $x'(t) = x^3 + 2\,x^2 + x$

1c) $x'(t) = \sin(x)$.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c.

There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

2b) $x'(t) = x^3 + 2x^2 + x$

2c) $x'(t) = \sin(x)$.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

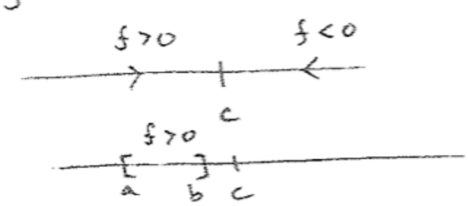
with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f)$: $- - - - 0 + + +$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable
$sign(f)$: $+ + + 0 - - - -$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is asymptotically stable
$sign(f)$: $+ + + 0 + + + +$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable (half stable)
$sign(f)$: $- - - - 0 - - - -$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ■

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic: $P'(t) = -a\,P^2 + b\,P$

Doomsday-extinction: $Q'(t) = a\,Q^2 - b\,Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M).$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M - P)$$

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q - M)\,.$$

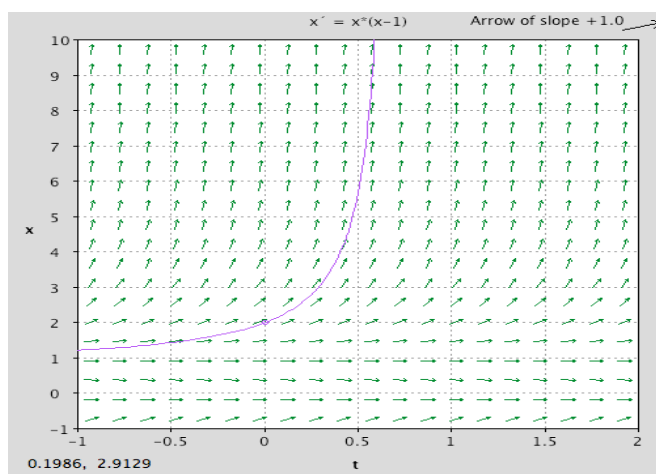
Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}\,.$$

Exercise 5: Use your formula from the previous exercise or work the separable DE from scratch, to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x - 1)$$
$$x(0) = 2\,.$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)



Math 2280-001
Fri Jan 25
2.2 Autonomous differential equations, with applications; 2.3 improved velocity models

Announcements:

Warm-up Exercise:

- Recall that on Wednesday we discussed the following important concepts:
  - * Autonomous first order DE
  - * equilibrium solutions for autonomous DE's
  - * stability at equilibrium points.

<u>Further application:</u> (related to parts of a "yeast bioreactor" homework problem for next week) harvesting a logistic population...text p.89-91 (or, why do fisheries sometimes seem to die out "suddenly"?) Consider the DE

$$P'(t) = a\,P - b\,P^2 - h\,.$$

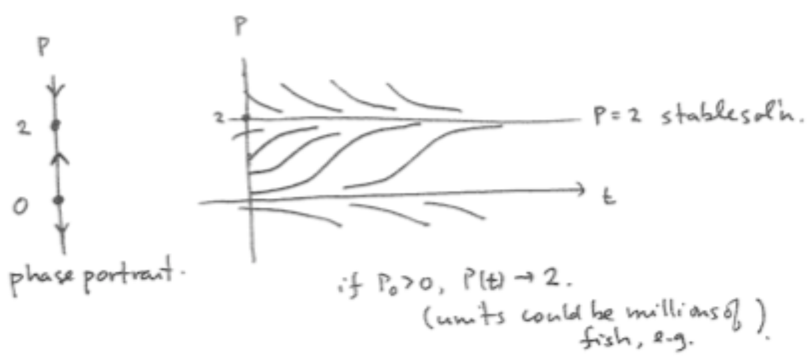
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity take $a=2,\ b=1$

<u>Case 0</u> no harvesting

$$P'(t) = 2P - P^2 = P(2-P)$$



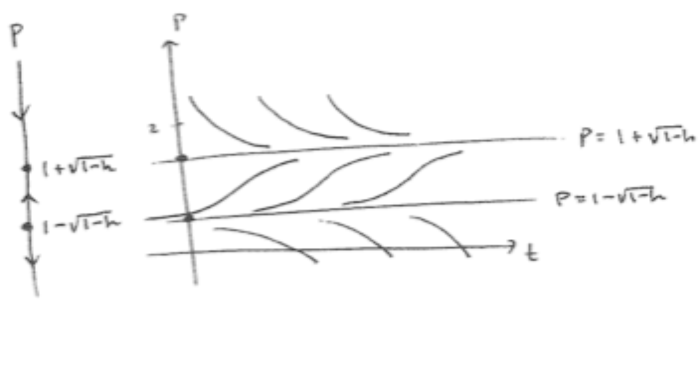
with harvesting:

$$P'(t) = 2P - P^2 - h = -(P^2-2P+h) = -(P-P_1)(P-P_2)$$

$$P_1, P_2 = \frac{2\pm\sqrt{4-4h}}{2} = 1\pm\sqrt{1-h}$$

<u>Case 1</u>: subcritical harvesting

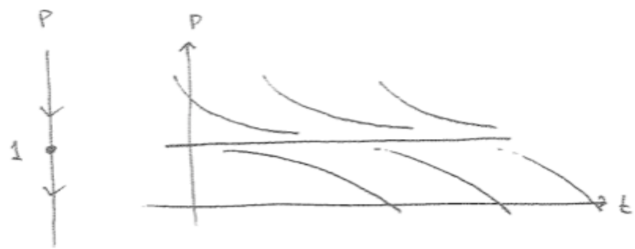
$0<h<1$



Case 2 Critical harvesting

$h = 1$

$P'(t) = -(P-1)^2$



Case 3 Over harvesting

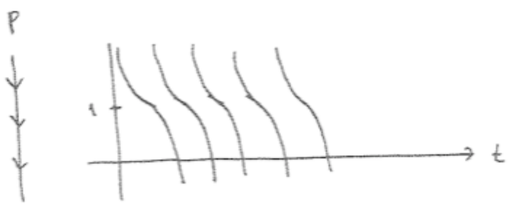
$h > 1$

complex roots.

$P'(t) = -(P^2 - 2P + h)$

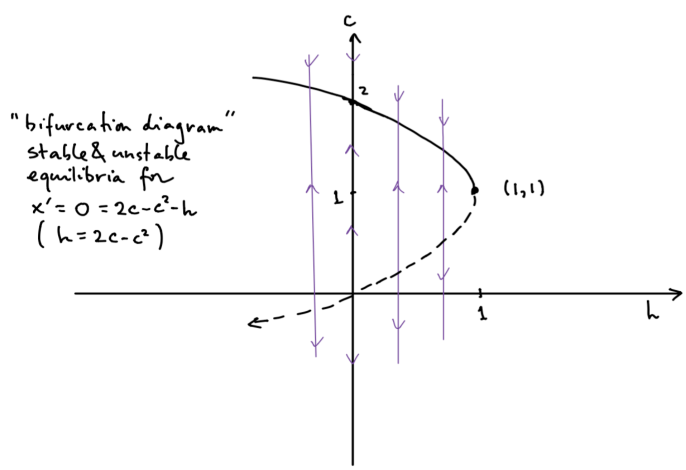
$= -[(P-1)^2 + (h-1)]$

$< 0.$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h . The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h . What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.



2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

$$\text{position } x(t) \text{ (or } y(t)\text{)},$$
$$\text{velocity } x'(t) = v(t), \text{ and}$$
$$\text{acceleration } x''(t) = v'(t) = a(t)$$

We have Newton's 2^{nd} law

$$m\, v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\, t + v_0$$
$$x(t) = \frac{1}{2}\alpha\, t^2 + v_0\, t + x_0 .$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\, v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\, |v|^p \ , 1 \le p \le 2 .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\,v'(t) = m\,\alpha - k\,v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = F_R(0) + F_R{}'(0)\,v + \frac{1}{2!}F_R{}''(0)v^2 + \ldots$$

$F_R(0) = 0$ and for small enough v the higher order terms might be negligable compared to the linear term, so

$$F_R(v) \approx F_R{}'(0)\,v \approx -k\,v \;.$$

We write $-k\,v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

Exercise 1: Let's rewrite the linear drag model

$$m\,v'(t) = m\,\alpha - k\,v$$

as

$$v'(t) = \alpha - \rho\,v$$

where the $\rho = \frac{k}{m}$. Now construct the phase diagram for v . (Hint: there is one critical value for v.)

The value of the constant velocity solution is called the *terminal velocity,* which makes good sense when you think about the underlying physics and phase diagram.

• $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number" Accounting for the fact that the resistance opposes direction of motion we get

$$m\, v'(t) = m\, \alpha - k\, v^2 \quad \text{if } v > 0$$

$$m\, v'(t) = m\, \alpha + k\, v^2 \quad \text{if } v < 0.$$

Do you understandt the sign of the drag terms in these two cases?

http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity

Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

$$v'(t) = \alpha - \rho\, v^2 \quad \text{if } v > 0$$
$$v'(t) = \alpha + \rho\, v^2 \quad \text{if } v < 0.$$

Exercise 2) Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction.

$$v'(t) = -g - \rho\, v^2 \quad \text{if } v > 0$$
$$v'(t) = -g + \rho\, v^2 \quad \text{if } v < 0.$$

Draw the phase diagrams. Note that each diagram contains a half line of v-values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \le 0$. Is there a terminal velocity?

How would you set up and get started on finding the solutions to these two differential equations? A couple of your homework problems are related to this quadratic drag model.

Exercise 3a Returning to the linear drag model and with gravity. Solve the IVP

$$v'(t) = \alpha - \rho\, v$$
$$v(0) = v_0$$

and verify that your solutions are consistent with the phase diagram analysis two pages back. (This is, once again, our friend the first order constant coefficient linear differential equation.)

3b integrate the velocity function above to find a formula for the position function $y(t)$. Write $y(0) = y_0$.

Comparison of Calc 1 constant acceleration vs. linear drag acceleration model:

We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial velocity of 49 $\frac{m}{s}$. (That initial velocity is chosen because its numerical value is 5 times the numerical value of $g = 9.8\ \frac{m}{s^2}$, which simplifies some of the computations.) In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of coupled differential equations for the horizontal and vertical motions, if you didn't shoot the bolt straight up. So we're shooting it straight up.

No drag:

$$v'(t) = -g \approx -9.8\ \frac{m}{s^2}$$

$$v(t) = -g\,t + v_0 = -g\,t + 5\,g = g\cdot(-t+5)\qquad \frac{m}{s}$$

$$x(t) = -\frac{1}{2}g\,t^2 + v_0\,t + x_0 = -\frac{1}{2}g\,t^2 + 5\,g\,t = g\,t\left(-\frac{1}{2}t + 5\right)\qquad m$$

So our deadbolt goes up for 5 seconds, then drops for 5 seconds until it hits the ground. Its maximum height is given by

$$x(5) = \frac{g\cdot 5\cdot 5}{2} = 122.5\ m$$

Linear drag: The same deadbolt, with the same initial velocity with numerical value $5\,g = 49\ \frac{m}{s}$. We're told that our deadbolt has a measured terminal velocity of $v_\tau = -245\ \frac{m}{s}$ which is the numerical value of $-25\,g$. The initial value problem for velocity is

$$v'(t) = -g - \rho\,v$$
$$v(0) = v_0 = 25\,g = 245.$$

So, in these letters the terminal velocity is (easily recoverable by setting $v'(t) = 0$) and is given by

$$v_\tau = -\frac{g}{\rho} = -25\,g \Rightarrow \rho = .04\,.$$

So, from our earlier work: Substituting $\alpha = -g$ into the formulas for terminal velocity, velocity, and height:

$$v(t) = v_\tau + \left(v_0 - v_\tau\right) e^{-\rho t} = -245 + 294\,e^{-.04\,t}\,.$$
$$y(t) = -245\,t + \frac{294}{.04}\left(1 - e^{-.04\,t}\right).$$

The maximum height occurs when $v(t) = 0$,

$$-245 + 294\, e^{-.04\, t} = 0$$

which yields $t = 4.56$ sec:

> $-\dfrac{\ln\left(\dfrac{245.}{294.}\right)}{.04};$

$$4.558038920 \qquad (6)$$

And the maximum height is 108.3 m:

> $y := t \rightarrow -245.\cdot t + \dfrac{294}{.04}\left(1 - e^{-.04\cdot t}\right);$
> $y(4.558038920);$

$$y := t \rightarrow (-1)\cdot 245.\, t + \frac{294\left(1 - e^{(-1)\cdot 0.04\, t}\right)}{0.04}$$

$$108.280465 \qquad (7)$$

So the drag caused the deadbolt to stop going up sooner ($t = 4.56$ vs. $t = 5$ sec) and to not get as high (108.3 *vs* 122.5 *m*). This makes sense. It's also interesting what happens on the way down - the drag makes the descent longer than the ascent: 4.85 seconds on the descent, vs. 4.56 on the ascent.

> $solve(y(t) = 0, t);$

$$9.410949931, 0. \qquad (8)$$

> $9.411 - 4.558;$

$$4.853 \qquad (9)$$

IMPORTANT to note that we needed to use a "solve" command (or something sophisticated like Newton's method) to find when the deadbolt landed. You cannot isolate the t algebraically when trying to solve $y(t) = 0$ for t. This situation will also happen in some of your homework problems this week.

$$-245.\cdot t + \frac{294}{.04}\left(1 - e^{-.04\cdot t}\right) = 0$$

picture:

```
> z := t→49 t − 4.9·t^2 :
  with(plots) :
  plot1 := plot(z(t), t = 0 ..10, color = green) :
  plot2 := plot(y(t), t = 0 ..9.4110, color = blue) :
  display({plot1, plot2}, title = `comparison of linear drag vs no drag models`);
```