Math 2280-2
Week 2, Jan 14-18: sections 1.3-1.5, 2.1.
Mon Jan 14
Finish discussion of existence-uniqueness theorem from 1.3; Toricelli's law application from 1.4.

Announcements:

Warm-up Exercise:

Review of last week:

We understand what it means for functions $y(x)$ to solve a differential equation

$$y' = f(x, y)$$

and/or an initial value problem on some interval I with $x_0 \in I$:

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

We know how to find implicit and possibly explicit solutions to *separable* differential equations

$$y'(x) = f(x)g(y)$$

which extends the special case of direct antidifferentiation

$$y'(x) = f(x).$$

We understand the connection between slope fields for differential equations and graphs of solutions to initial value problems.

Because of geometric intuition based on slope fields, we expect each initial value problem for a reasonable first order differential equation to have one and only one solution, at least defined on some interval containing the intial variable value. On Friday we saw that this isn't actually always true, but there is a Theorem that explains the situation...

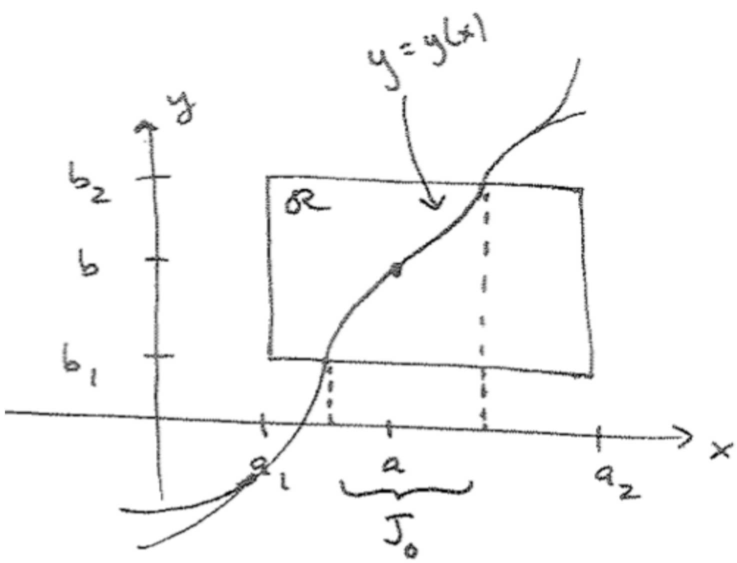
Here's what's going on (stated in 1.3 page 22 of text as *Theorem 1*; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \le x \le a_2, b_1 \le y \le b_2$ that *you* specify in the $x-y$ plane.

- <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the *y*-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 1</u> On the next page - which is our last completed page from Friday - explain how existence-uniqueness theorem applies for that initial value problem

From Friday...

Exercise 3a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(0) = 0$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

3c) Sketch some of these singular solutions onto the slope field below.

$\frac{dy}{dx} = f(x)\,g(y)$

$\frac{dy}{g(y)} = f(x)\,dx$ Assuming $g(y) \neq 0$

If $g(y^*) = 0$ then $y(x) = y^*$ is a sol'n. (y^* const)

$y'(x) = 0$

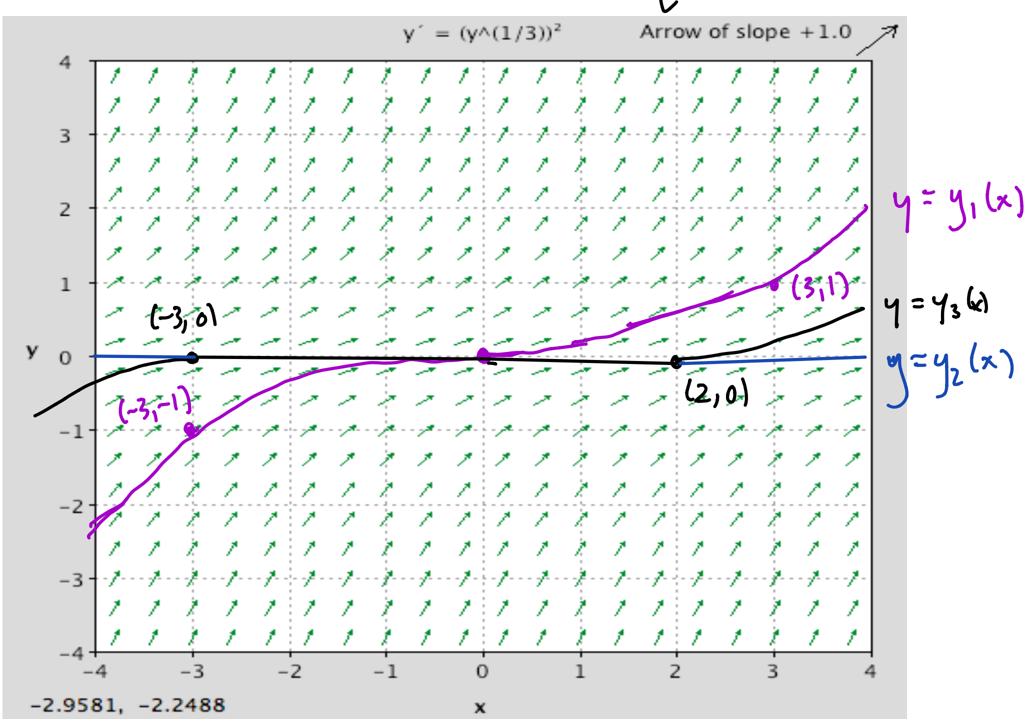
$f(x)g(y) = f(x)g(y^*) = f(x)\cdot 0 = 0$

warmup 3a). We got sol'ns to the DE $y(x) = \frac{1}{27}(x+C)^3$

so for IVP $y_1(x) = \frac{1}{27}x^3$

3b) also noticed $y_2(x) \equiv 0$ solves IVP. keyed into that via sep. of variables

∞'ly many sol'ns $y_3(x) = \begin{cases} 0 & -3 \le x \le 2 \\ \frac{1}{27}(x-2)^3 & x \geq 2 \\ \frac{1}{27}(x+3)^3 & x < -3 \end{cases}$ "2" was arbitrary, "-3" was arbitrary



Exercise 2 (A slight variation on the preceding one. Also, one of your homework problems is similar.) Consider the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(3) = 8.$$

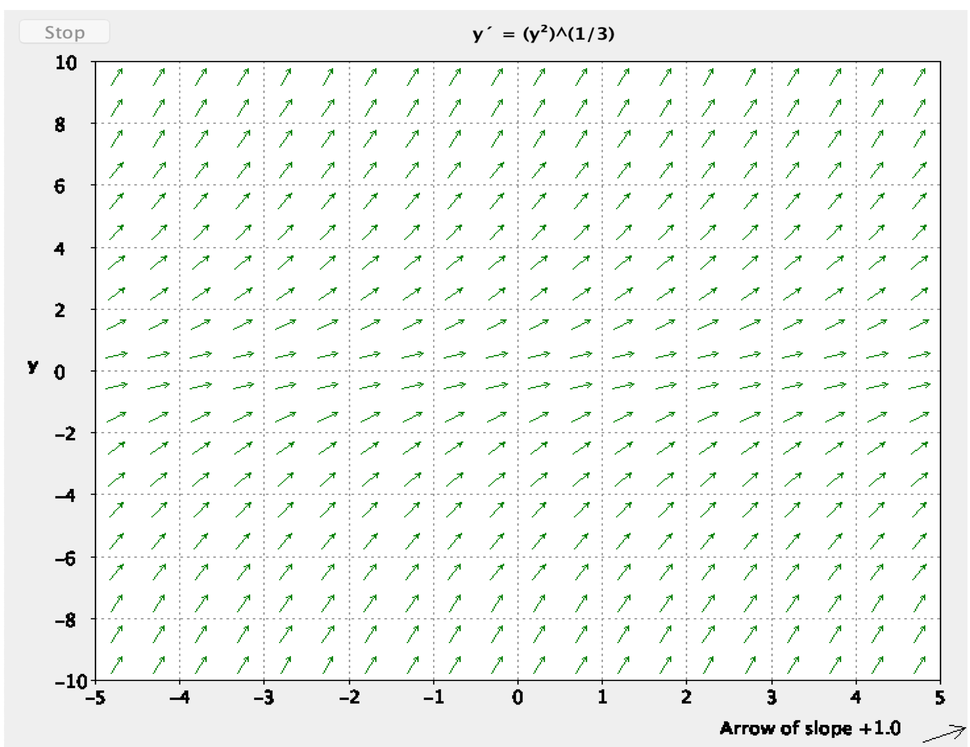
a) Does the IVP above have a unique solution on some interval containing $x_0 = 3$, according to the existence-uniquenss theorem?

b) Find the IVP solution above, using separation of variables solutions that we found on Friday

$$y(x) = \frac{1}{27}(x+C)^3 = \left(\frac{x+C}{3}\right)^3$$

c) What is the largest interval on which it is the unique solution? Sketch below! What's the biggest rectangle $\mathcal{R}$ that you can specify for uniqueness?

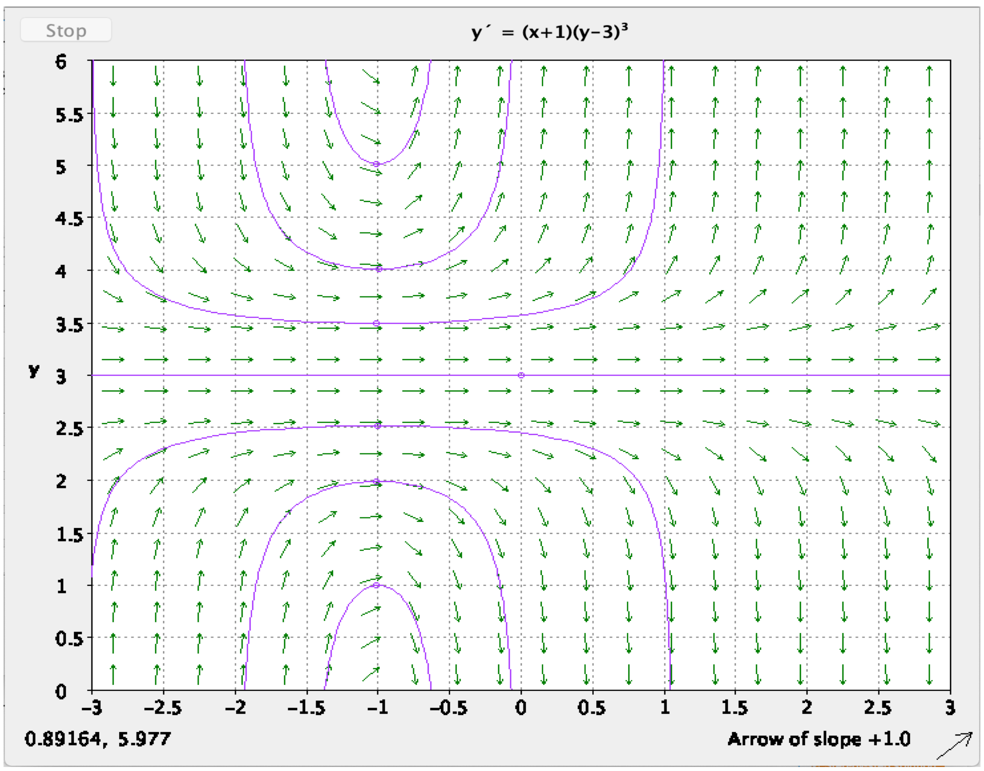
d) What happens when you solve this IVP numerically with dfield? I'll demo dfield, since I'm asking you to use it for your homework. Wolfram alpha makes slope fields, but they're pretty low quality.



Exercise 3: Do the initial value problems below always have unique solutions? Would you be able find formulas for them? (Notice two of these are NOT separable differential equations.) Can technology find formulas for the solution functions?

a)

$$y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$



I used Maple - you'd get the same results with Wolfram alpha. Which solution did technology miss?

```
> with(DEtools) :
> dsolve(y'(x) = (x + 1)·(y(x) − 3)^2, y(x));
```

$$y(x) = \frac{3x^2 + 6\_C1 + 6x - 2}{x^2 + 2\_C1 + 2x} \quad \textbf{(1)}$$

```
>
```

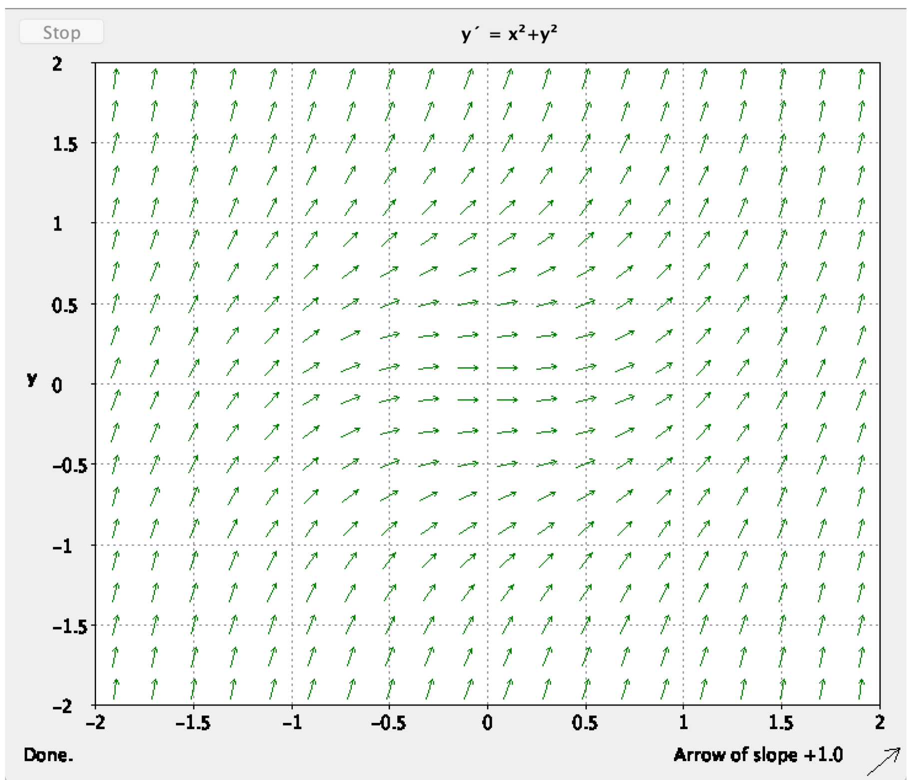
b)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^2 + y(x)^2, y(x)\right);$

$$y(x) = \frac{\left(-\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\_C1 - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\right)x}{\_C1\ \mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)} \quad \textbf{(2)}$$

>

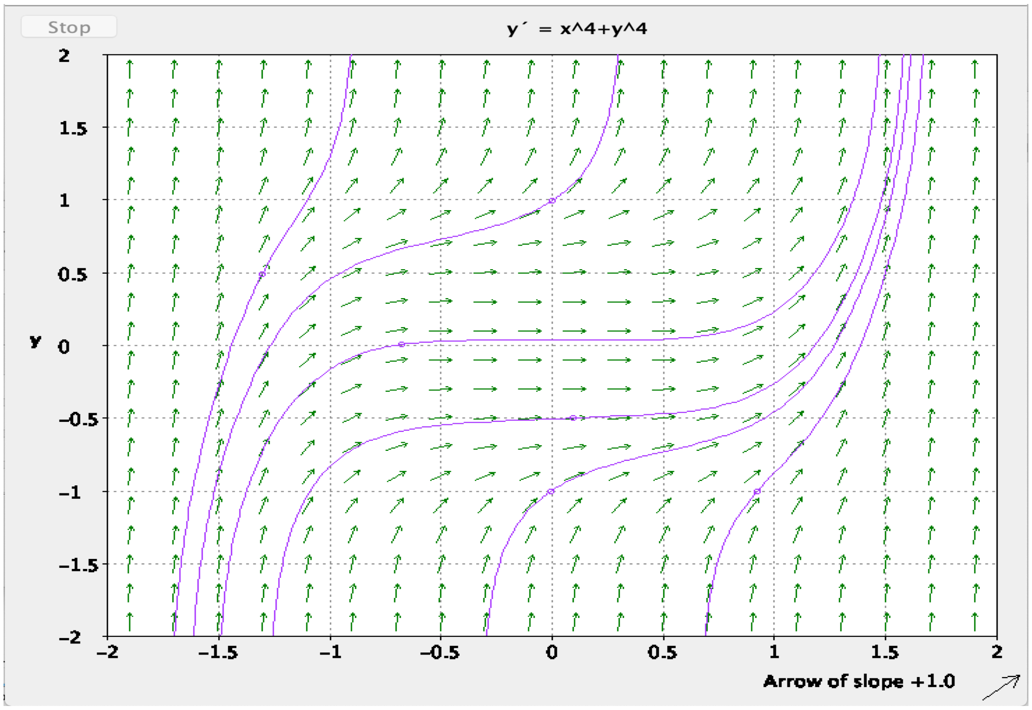


c)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^4 + y(x)^4, y(x)\right);$

>



For your section 1.2 and 1.4 homework this week I assigned a selection of application problems. Some applications will be familiar to you from previous courses, e.g. exponential growth and Newton's Law of cooling, velocity-acceleration problems. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

Toricelli's Law, for draining water tanks. Refer to the figure below.

Exercise 1:

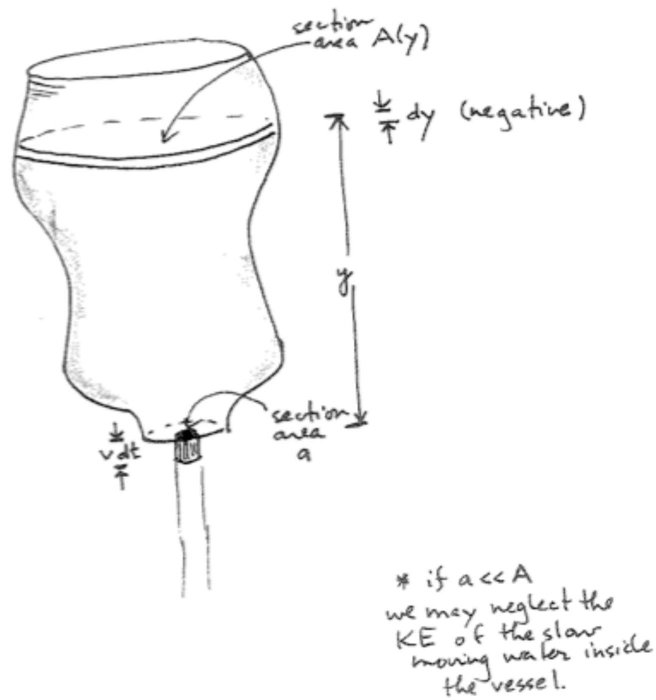
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity ($a << A$) . Equate the lost potential energy from the top in time *dt* to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed *v* with which the water exits the tank is given by

$$v = \sqrt{2gy}$$

when the water depth above the hole is $y(t)$ (and *g* is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2g})\ .$$



Experiment fun!   I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k than on previous page).

Exercise 4a) Solve the differential equation and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank.

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = 1$$

Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

and the inverse function $t = t(y)$ is given by

$$t = \frac{2}{k}\left(1 - \sqrt{y}\right)$$

(until the tank runs empty).

Exercise 4b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let t_1 be the time it takes the the water to go from height 1 (full) to height 0.5 (half empty). Let t_2 be the time it takes for the water to go from height 1 (full) to height 0.0 (empty). Show that

$$t_2 = \frac{1}{1 - \sqrt{.5}}\, t_1 \approx 3.41\, t_1.$$

<u>Experiment!</u> We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 : # that should be enough significant digits
```

$$> \frac{1}{1-\mathrm{sqrt}(.5)}; \text{\# the factor from previous page}$$

$$3.4143 \qquad (3)$$

```
> Thalf := 35;   # seconds to half-empty canteen in a previous test.
  Tpredict := 3.4143·Thalf; #prediction
```

$$Thalf := 35$$

$$Tpredict := 119.50 \qquad (4)$$

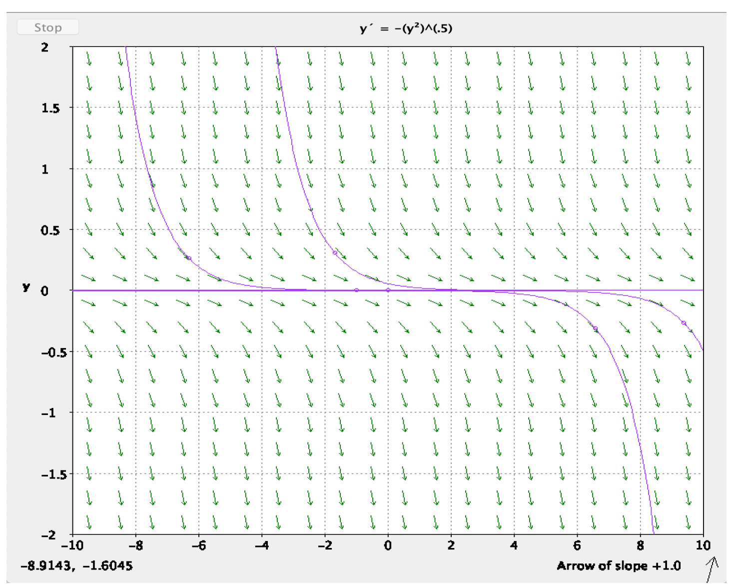
`>`

What are possible defects in our model?

What does the existence-uniqueness theorem say about solutions to IVP's for this DE when the initial height is zero? Does this make sense? (Notice, I extended the square root function from positive to negative values by taking the square root of the absolute value, so that the existence theorem applies.)

$$\frac{dy}{dt} = -k\,|y|^{\frac{1}{2}}$$

$$y(0) = 0$$



Tues Jan 15
1.5 linear first order differential equations.

Announcements:

Warm-up Exercise:

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be (re)written as

$$y' + P(x)y = Q(x)$$

Exercise 1 Classify the differential equations for $y(x)$ below as linear, separable, both, or neither. Justify your answers by rewriting the DE (if necessary) so that it is in the standard format for linear or separable differential equations

a) $y' = -2y + 4x^2$

b) $y' = x - y^2 + 1$

c) $y' = y - x$

d) $y' = \dfrac{6x - 3xy}{x^2 + 1}$

e) $y' = x^2 + y^2$

f) $y' = x^2 e^{x^3}$

Remark: We call these *linear* differential equations because we can rewrite them as

$$L(y) = Q(x)$$

where

$$L(y) := y' + P(x)y$$

is a *linear transformation* (from 2270) !!!

Exercise 2) (This is one of your homework exercises....)

Consider exercise 1c above, written in "linear form":

$$y' - y = -x$$

Multiply both sides by the never-zero exponential function e^{-x} to get an equivalent differential equation - in the sense that solutions to the first DE are also solutions to the second DE, and vise-verse. Why did we choose e^{-x} ? It has nothing to do with the right side of this DE, and everything to do with the left side. That's explained on the next page!

$$e^{-x}(y' - y) = -x\,e^{-x}.$$

Now, if you're able to cleverly recognize the expression on the left as the derivative of a product, you'll be able to antidifferentiate both sides with respect to x, and find all solutions!

Recall that the method for solving separable differential equations via differentials was actually using the differentiation chain rule "backwards" to antidifferentiate and find the solution functions. *The algorithm for solving linear DEs is a method to use the differentiation product rule backwards, after replacing the differential equation with its multiple by a non-zero "integrating factor" exponential function.* Here's how it goes in general!

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be *any* antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx}Q(x)$$

This makes the left side a derivative (check via product rule):

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)\ .$$

So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx \quad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

Remark: If we abbreviate the function $e^{\int P(x)dx}$ by renaming it $G(x)$, then the formula for the solution $y(x)$ to the first order DE above is

$$y(x) = \frac{1}{G(x)}\int e^{\int P(x)dx}Q(x)dx + \frac{C}{G(x)}\ .$$

If x_0 is a point in any interval I for which the functions $P(x)$, $Q(x)$ are continuous, then $G(x)$ is positive and differentiable, and the formula for $y(x)$ yields a differentiable solution to the DE. By adjusting C to solve the IVP $y(x_0) = y_0$, we get a solution to the DE IVP on the entire interval. And, rewriting the DE as

$$y' = -P(x)y + Q(x)$$

we see that the existence-uniqueness theorem implies this is actually the only solution to the IVP on the interval (since $f(x, y)$ and $\frac{\partial}{\partial y}f(x, y) = P(x)$ are both continuous). These facts would not necessarily be true for separable DE's...and we've seen how separable DE solutions may not exist or be unique on arbitrarily large intervals.

Exercise 3 Verify that our work in Exercise 2 was following this recipe.

Exercise 4: Find all solutions to the linear (and separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

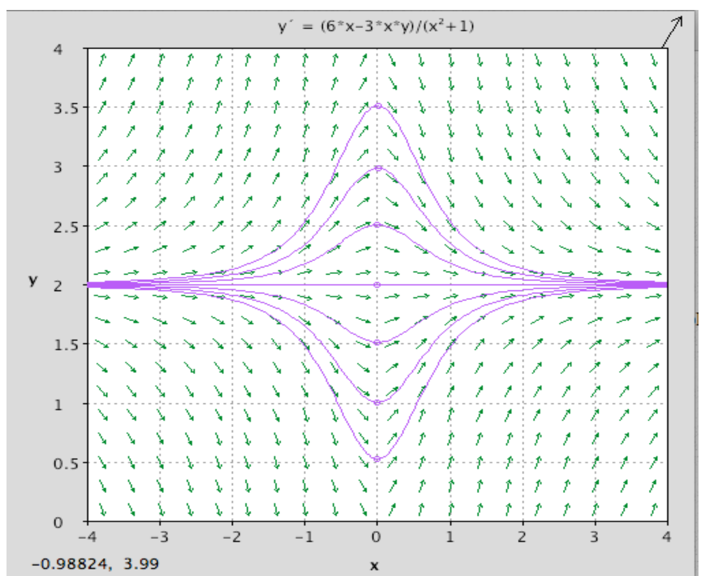
Hint: as you can verify below via Wolfram alpha, the general solution is $y(x) = 2 + C\left(x^2 + 1\right)^{-\frac{3}{2}}$. Notice that the right side of this differential equation satisfies the existence-uniqueness theorem for the rectangle which is all of $\mathbb{R}^2$, and our unique solutions exist on all of $\mathbb{R}$, $-\infty < x < \infty$ (in contrast to what can happen for general separable differential equations).

Input:

$$y'(x) = \frac{6x - 3x\,y(x)}{x^2 + 1}$$

Differential equation solution:

$$y(x) = \frac{c_1}{(x^2 + 1)^{3/2}} + 2$$



Wed Jan 16
1.5 applications of linear differential equations

Announcements:

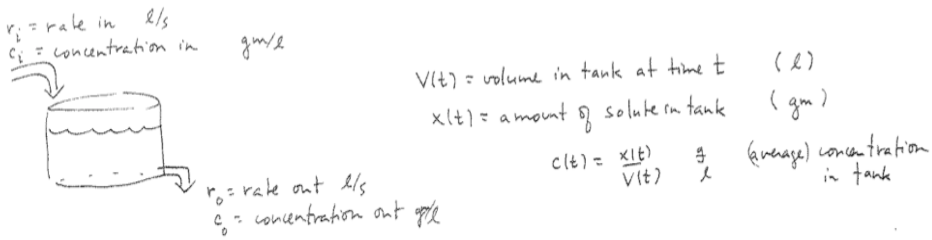
Warm-up Exercise:

An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o, and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t, given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}.$$

See the diagram below.



Exercise 1: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants.

Exercise 2: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t} .$$

Exercise 3 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \frac{km^3}{y}$

b) Solve the IVP, and then answer the question.

Fri Jan 16
2.1 improved population models

Announcements:

Warm-up Exercise:

2.1: Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per $person$)

$D(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{D(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per $person$)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - D(t)$$
$$P'(t) = (\beta(t) - \delta(t))P(t).$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP.$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P))P$$
$$= ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P.$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$$P' = (b - aP)P$$
$$P' = bP - aP^2 \text{, or equivalently}$$
$$P' = aP\left(\frac{b}{a} - P\right) = kP(M - P).$$

$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1a): Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$. Notice that the "isoclines" (curves where the slope function is constant) are horizontal lines

$$P' = k\,P(M - P)$$

Also note that there are two constant ("equilibrium") solutions. What are they?

b) Sketch the slope field and apparent solutions graphs in a qualitatively accurate way. We'll also include the 1-dimensional "phase portrait" associated to these slope fields.

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? Note that by the existence-uniqueness theorem, different solution graphs may never touch each other, so the time-varying solution graphs never touch the horizontal graph asymptotes.

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}\,.$$

Solution (we will work this out step by step in class, using the fact that the logistic DE is separable. It is not linear!!):

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} .$$

Notice that because $\lim_{t \to \infty} e^{-Mkt} = 0$,

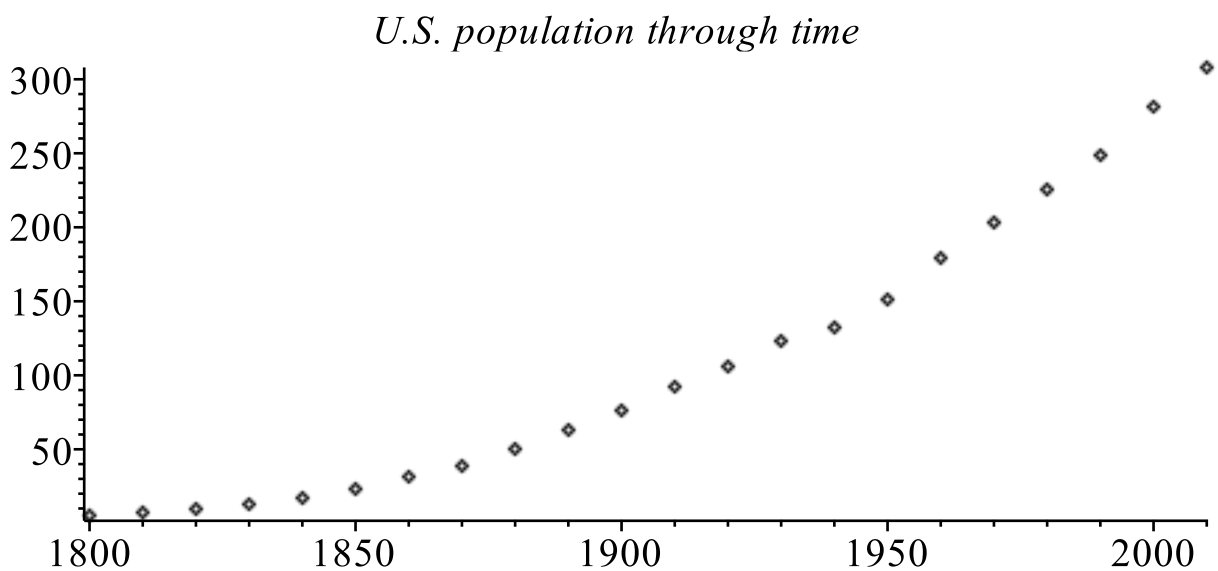
$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \quad (5)$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \quad (6)$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, e^{-M\, k\, t}} \quad (7)$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12, k = 0.00016772\} \quad (8)$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
    #equation (11) page 84.
```

$$M := 188.12$$

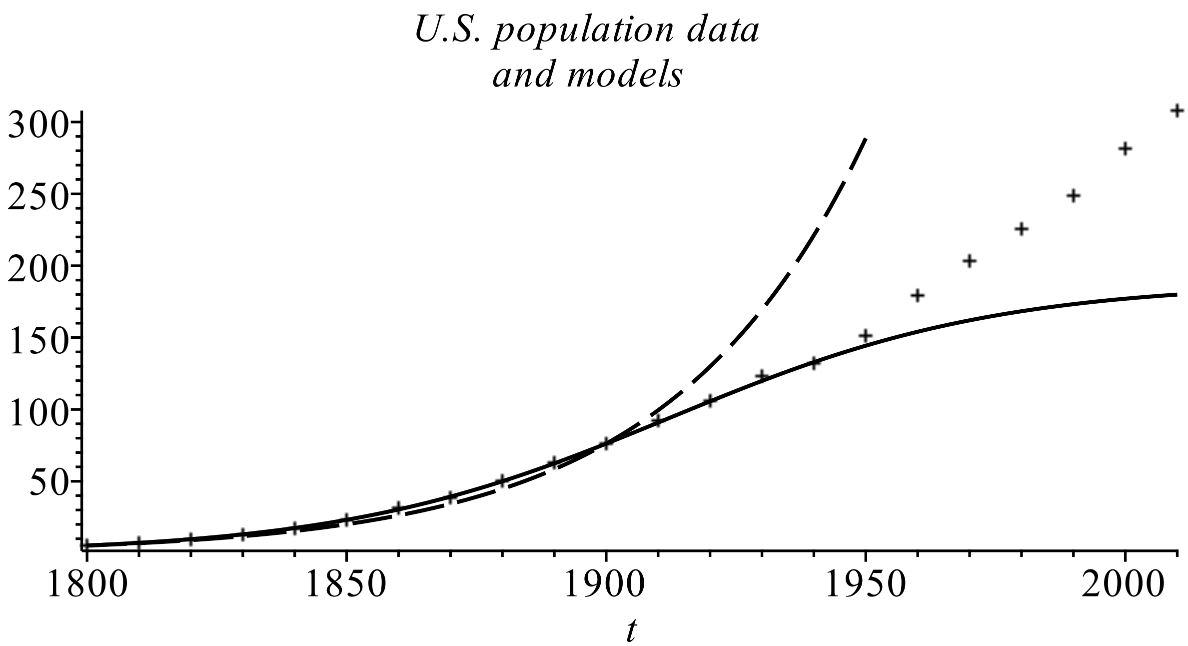
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, e^{-0.031551\, t}} \quad (9)$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources.

```
> plot1 := plot(P1(t-1800), t = 1800..1950, color = black, linestyle = 3) :
  #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t-1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```



>

Any ideas on why the logistic model begins to fail (with our parameters) around 1950?