Pick up
- syllabus
- week 1 outline
- HW #1

Pick up & return
- sheet of paper ~ tell me briefly who you are & why you're interested in Math 2280.

Math 2280-002 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes are for sections 1.1-1.3, and part of 1.4.

Monday January 7

- Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2280spring19

- Notice that there our first homework assignment is due next Wednesday, but that we will already have a quiz this Wednesday.

Then, let's begin!

Use notes as outline
work along in blank areas as we discuss topics

Syllabus for Math 2280-002 Differential Equations

Spring 2019

Instructor: Professor Nick Korevaar
email: korevaar@math.utah.edu
office: LCB 204, 801.581.7318
office hours: TBA
Lecture: MTWF 12:55-1:45 p.m. in LCB 219

Course websites

Lecture notes and homework assignments will be posted on our public home page.

http://www.math.utah.edu/~korevaar/2280spring19

There are blank spaces in the notes where we will work out examples and fill in details together. Research has shown that class attendance with active participation - including individual and collaborative problem solving, and writing notes by hand - are effective ways to learn class material for almost everyone. Passively watching a lecture without active participation is not usually very effective. Class notes will be posted on our public page, and I plan to bring weekly packets to class for you to use. Beyond what is outlined in the notes, there will be additional class discussion related to homework and other problems. Grades and exam material will be posted on our CANVAS course page; access via Campus Information Systems.

Textbook *Differential Equations and Boundary Value Problems, Computing and Modeling, 5th edition*, by Edwards and Penney. ISBN=978-0-321-79698-1.

Final Exam logistics: Tuesday April 30, 1:00 -3:00 p.m., in our classroom LCB 219. This is the University scheduled time and location.

Catalog description for Math 2280: Linear and nonlinear differential equations and systems of equations, with applications. Matrix exponential, fundamental solution matrix, phase-space and portraits, stability, initial- and boundary-value problems, introduction to partial differential equations. Requires familiarity with linear algebra. Includes theoretical and computer lab components.

matlab // wolfram alpha

Course overview: Math 2280 is an introduction to ordinary differential equations, and how they are used to model problems arising in engineering and science. It is the second semester of the year long sequence 2270-2280, which is an in-depth introduction to linear mathematics. Along with Calculus, the linear algebra that you learned in Math 2270 will provide a surprising amount of the conceptual and computational framework for our discussions in Math 2280. This will become more apparent as the course progresses.

Prerequisites: Linear Algebra, Math 2270, and its prerequisite courses of single and multi-variable calculus.

Students with disabilities:

The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in the class, reasonable prior notice needs to be given to the Center for Disability Services, 162 Olpin Union Building, 581-5020. CDS will work with you and the instructor to make arrangements for accommodations. All information in this course can be made available in alternative format with prior notification to the Center for Disability Services.

Grading

Math 2270-002 is graded on a curve. By this I mean that the final grading scale may end up lower than the usual 90/80/70% cut-offs. note: In order to receive a grade of at least C in the course you must earn a grade of at least C on the final exam. Typical final grade distributions in Math 2280 are roughly 40% each for As and Bs, and 20% with lower grades. Individual classes may vary. Details about the content of each assignment type, and how much they count towards your final grade are as follows:

- Homework (30%): There will be one homework assignment each week. Homework problems will be posted on our public page, and homework assignments will be due in class on Wednesdays. Homework assignments must be stapled. Unstapled assignments will not receive credit. I understand that sometimes homework cannot be completed on time due to circumstances beyond your control. To account for this, each student will be allowed to turn in two late homework assignments throughout the course of the semester. These assignments cannot be turned in more than one week late, and must be turned in on a Wednesday with the next homework assignment. You do not need to tell me the reason why your homework assignment is late. Homework will be a mixture of problems from the text and custom problems, and will vary from computational practice to modeling and more conceptual questions. There will be applications that require technology to complete. We will make use of Matlab for some of these applications.

- Quizzes (10%): At the end of most Wednesday classes, a short 1-2 problem quiz will be given, taking roughly 10 minutes to do. The quiz will cover relevant topics from the weeks lectures and homework. Your lowest two quiz scores will be dropped. There are no makeup quizzes. You will be allowed and encouraged to work together on these quizzes. starting this week

- Midterm exams (30%): Two class-length midterm exams will be given, On Friday February 15 and Friday March 29. I will schedule a room for review on the Thursday before each midterm, at our regular class time of 12:55-1:45 p.m. No midterm scores are dropped.

- Final exam (30%): A two-hour comprehensive exam will be given at the end of the semester. As with the midterms, a practice final will be posted. Please check the final exam time, which is the official University scheduled time. It is your responsibility to make yourself available for that time, so make any arrangements (e.g., with your employer) as early as possible.

Strategies for success:

- Attend and participate in class regularly.

- Read or at least scan the relevant text book sections and lecture note outlines *before* you attend class.

- Ask questions and become involved.

- Plan to do homework daily; try homework on the same day that the material is covered in lecture; do not wait until just before homework is due to begin serious work.

- Form study groups with other students.

(read on your own - we'll revisit this page
at the end of the course to see how we did)

Learning objectives for Math 2280: The goal of Math 2280 is to master the basic tools and problem solving techniques important in differential equations, as well as to understand the larger conceptual framework that ties these ideas together. Specific goals include:

- Be able to model dynamical systems that arise in math, science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental analysis, Newtons laws and conservation of energy.
- Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering.
- Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and numerical approximations to initial value problems via slope fields. Understand phase diagram analysis for autonomous first order differential equations.
- Apply vector space concepts from linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear differential equations and linear systems of differential equations.
- Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand solutions to the basic unforced and forced mechanical and electrical oscillation problems.
- Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, electrical circuits, and related problems.
- Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.
- Be able to apply matrix algebra concepts related to eigenvalues, eigenvectors and matrix diagonalization, in order to find the general solution space to first and second order constant coefficient homogeneous linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.
- Learn how to work with matrix exponentials and other fundamental matrix solutions, as tools for understanding linear systems of differential equations with constant coefficients.
- Understand and be able to use linearization as a technique to understand the behavior of nonlinear dynamical systems near equilibrium solutions. Apply these techniques to autonomous systems of two first order differential equations, including interacting populations and systems arising from non-linear forced oscillation problems.
- Learn about Fourier series, and use them as an "infinite superposition" tool to study forced oscillation problems.
- Learn how to find solutions to Laplace's equation, the heat equation and the wave equation using separation of variables, together with Fourier series and superposition.
- Develop your abilities to model dynamical systems with differential equations, and to understand solutions analytically and with technology and software such as Matlab and internet-based tools as appropriate.

Week-by-Week Topics Plan

Topic schedule is subject to slight modifications as the course progresses, but exam dates are fixed.

Week 1: 1.1-1.4; differential equations and mathematical models; slope fields and initial value problems; separable differential equations.

Week 2: 1.5, 2.1-2.2 linear differential equations; applications of separable equations to population models; phase diagram analysis.

Week 3: 2.3-2.6; improved acceleration-velocity models; numerical solutions to DEs.

Week 4: 3.1-3.3; higher order linear differential equations; homogeneous and non-homogeneous problems.

Week 5: 3.4-3.6; applications to unforced mechanical vibrations; non-homogeneous linear differential equations and applications to forced mechanical vibrations.

Week 6: 3.6-3.7; applications and review. **Midterm exam 1 on Friday February 15** covering material from weeks 1-6.

Week 7: 7.1-7.4 Laplace transform approach to linear differential equations.

Week 8: 4.1, 4.3, 5.1-5.2; first order systems of differential equations, numerical methods, analytic solution of homogeneous first order systems with eigenvalue and eigenvector computations, input-output models.

Week 9: 5.2-5.4; mechanical vibrations and second order systems.

Week 10: 5.4-5.7; multiple eigenvalue solutions, matrix exponentials, and applications

Week 11: continued and review **Midterm exam 2 on Friday March 29**, covering weeks 7-11 material.

Week 12: 6.1-6.4; non-linear systems of first order differential equations with applications to ecological models and nonlinear mechanical systems.

Week 13: 9.1-9.4; Fourier series, with application to forced oscillations.

Week 14: 9.5-9.7; introduction to partial differential equations.

Week 15: continued, and course review.

Week 16: **Final exam Tuesday April 30**, 1:00-3:00 p.m. in our classroom LCB 219. This is the University scheduled time and location.

returning to Monday notes...

Section 1.1 Introduction to differential equations

Definition An n^{th} *order differential equation* (DE) is any equation expressed in terms of an uspecified function $y = y(x)$ and its derivatives, for which the highest derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which after rearrangement can be written as

$$F\left(x, y(x), y'(x), y''(x), ... y^{(n)}(x)\right) = 0.$$

shorthand convention:

$$F\left(x, y, y', y'', y^{(n)}\right) = 0$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $\left(y''(x)\right)^2 + \sin(y(x)) = 0$. YES 2nd order

b) For $x = x(t)$, $x'(t) = 3\,x(t)(10 - x(t))$. YES [could rewrite as $x'(t) - 3x(t)(10 - x(t)) = 0$] 1st order

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$. YES (same as b)

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$. NO not an equation! (equations have equal signs)

e) For $y = y(x)$, $y' = y^2$. YES. 1st order. (the "2" is a power, not a derivative).

f) For $y = y(t)$, $y' = y^{(2)}$ is a 2nd order DE

Related definitions:

- A specified function or functions $y(x)$ *solve(s)* the *differential equation*

$$F\left(x, y, y', y'', \; y^{(n)}\right) = 0$$

on some interval I of $x-$values (or is a *solution function* for the differential equation) means that $y(x)$ makes the differential equation a true identity for all x in I.

Chapters 1-2 are about first order differential equations, algebraic and graphical representations of their solutions, and applications. For first order differential equations

$$F(x, y, y') = 0$$

we can often use algebra to solve for y' in order to get what we call the **standard form** for the first order DE:

$$y' = f(x, y) \; .$$

• If we want our solution function to a first order DE to also satisfy $y\left(x_0\right) = y_0$, and if our DE is written in standard form, then we say that we are studying an *initial value problem (IVP):*

$$\text{IVP} \; \begin{cases} y' = f(x, y) \\ y\left(x_0\right) = y_0 \, . \; \leftarrow \text{initial condition} \end{cases}$$

If we can find a solution function $y(x)$ to the DE satisfying the *initial condition* $y\left(x_0\right) = y_0$, then we say that $y(x)$ *solves* the initial value problem.

Exercise 2: Consider the differential equation $\frac{dy}{dx} = y^2$ from (1e). i.e. $y'(x) = (y(x))^2$

a) Show that functions $y(x) = \frac{1}{C - x}$ *solve* the DE (on any interval not containing the constant C). We'll see how we found these functions in part 2e, but we don't need that information to check whether or not they actually solve the differential equation.

b) Find the appropriate value of C to *solve the initial value problem*

$$\begin{cases} y' = y^2 \\ y(1) = 2 \, . \end{cases}$$

a) Does $y(x)$ make the DE a true identity?

$$y(x) = \frac{1}{C-x} = (C-x)^{-1}$$

then $y'(x) = +\frac{1}{(C-x)^2}$! LHS of DE

$$\frac{d}{dx}(C-x)^{-1} = -1(C-x)^{-2}(-1) = \frac{1}{(C-x)^2}$$

chain rule!

(alternate: quotient rule)

$$y(x)^2 = \left(\frac{1}{C-x}\right)^2 = \frac{1}{(C-x)^2} \quad \text{RHS of DE}$$

So for these functions the DE is a true identity (on any interval not containing $x = c$)

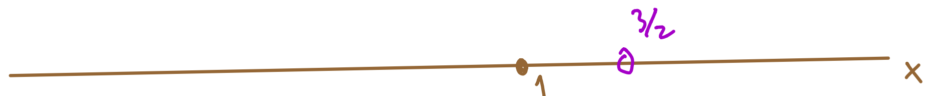
So, they are solutions

b) $y(x) = \dfrac{1}{C-x}$

want $y(1) = 2$: $\dfrac{1}{C-1} = 2$ solve for $C = 3/2$

$(C - 1 = \tfrac{1}{2})$

$$\boxed{y(x) = \frac{1}{3/2 - x}}$$



2c) What is the largest interval on which your solution to 2b is defined as a differentiable function? Why?

$(-\infty, 3/2)$

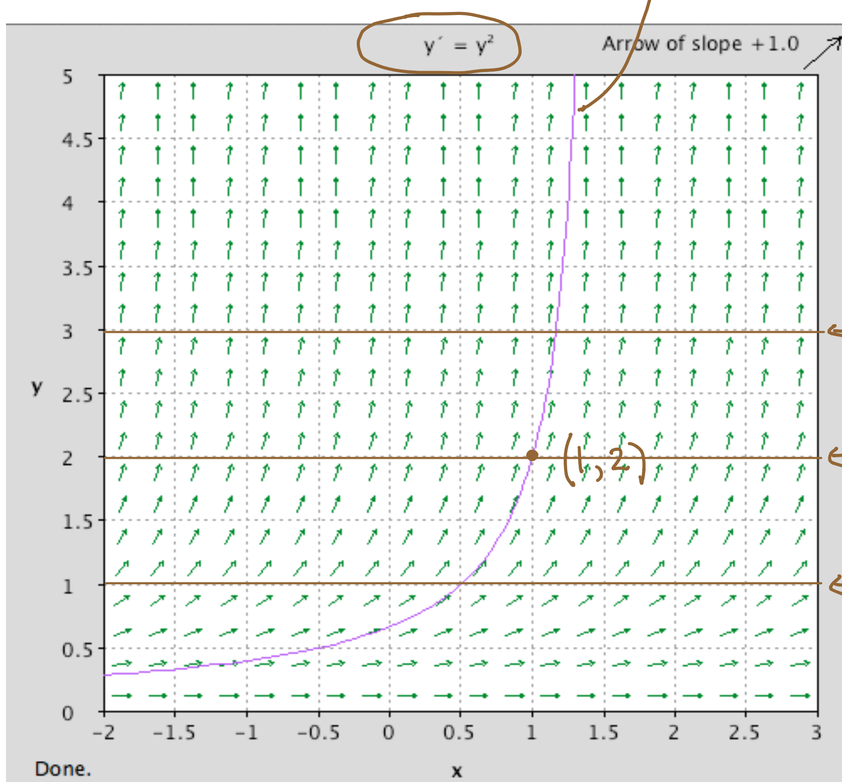
i.e. $-\infty < x < 3/2$

2d) Do you expect that there are any other solutions to the IVP in 2b (on the same interval)? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below, where the line segment slopes at points (x, y) have values y^2 (because solution graphs to our differential equation will have those slopes, according to the differential equation). This might give you some intuition about whether you expect more than one solution to the IVP.

We expect only one possible graph that is tangent to the slope field and that goes through the initial point (1, 2).

$y' = y^2$ $\qquad y = \dfrac{1}{3/2 - x}$

at points (x,y) slopes of green segments are given by "y^2" for graph of any soln to this DE, the graph will be tang. to slope field



all slopes are 9.

all slopes = 4

all slopes = 1

2e) How did someone find formulas for the solution functions in part 2a? Or how could we have found them if they weren't given to us?

$$\boxed{\frac{dy}{dx} = y^2}$$

Tuesday start here

Answer: They used the chain rule in reverse .... the systematic way of doing this is called "separation of variables", section 1.4, which we'll discuss in more detail tomorrow and which many of you discussed in a prerequisite Calculus class. Let's work the "chain rule in reverse" for this example in order to recall ...

1st try might be to antidifferentiate with respect to x

$$\frac{dy}{dx} = y(x)^2$$

don't know y(x) so can't antidiff RHS (right hand side)

2nd try: $\frac{y'(x)}{y(x)^2} = 1 \qquad (y(x) \neq 0)$

now $\int \frac{y'(x)}{y(x)^2}\,dx = \int 1\,dx$

let $u(x) = y(x)$

$du = y'(x)\,dx$

$\int \frac{du}{u^2}$

$= -\frac{1}{u} + C$

$$\boxed{= -\frac{1}{y(x)} + C = x + D}$$

$= -\frac{1}{y(x)} = x + E \qquad (E = D - C)$

So $y(x) = -\frac{1}{x+E} = \frac{1}{-x-E}$

we were told

$y(x) = \frac{1}{C-x}$

equivalent formulas

differentials shortcut (algorithm for separation of variables)

$$\frac{dy}{dx} = y^2$$

$$dy = y^2\,dx$$

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int 1 \cdot dx$$

$$\boxed{-\frac{1}{y} + C = x + D}$$

• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example, you've seen the *exponential growth/decay differential equation*, modeling situations in which *The rate of change of the quantity $P(t)$ is proportional to $P(t)$*:

$$P'(t) = k\,P(t)$$

with solutions

$$P(t) = P_0\,e^{k\,t}.$$

And you've seen the *constant acceleration* particle motion differential equation

$$y''(t) = a \quad \text{(constant)}$$

with solutions

$$y(t) = y_0 + v_0\,t + \frac{a}{2}t^2.$$

We'll see many more differential equations applications in this class. The general modeling paradigm and feedback loop is discussed in our text in section 1.1:

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4 Chapter 1 First-Order Differential Equations

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

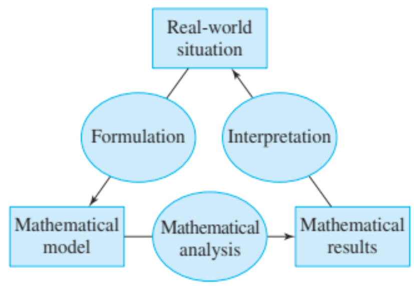


FIGURE 1.1.4. The process of mathematical modeling.

For example, the exponential growth model is effective for continuous compound interest in bank accounts, bacterial growth with no resource constraints, and radioactive decay (negative k). And the constant acceleration model is effective when there are no drag forces on the object and the underlying acceleration is close to a constant. But more sophisticated models are needed if the situation is less simple, e.g. the *logisitic* population model for populations with resource constraints, and particle acceleration models that need to take into account drag forces and non-constant background acceleration force. We will study such modifications in Chapter 2.

As a concrete prototype for how mathematical modeling works, consider:

to give credibility.

Exercise 3) *Newton's law of cooling* is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ is assumed to change at a rate proportional to ~~to~~ the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use the assumptions in the model above, to "derive" (i.e. explain) the differential equation for the $T(t)$ of the object being heated or cooled:

$$\frac{dT}{dt} = -k(T - A).$$

we used $-k$ so that we could take $k > 0$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

would be fine, except k would be < 0 so that $T'(t) < 0$ when $A < T$, $T'(t) > 0$ when $A > T$

c) Use the Newton's law of cooling model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 °F. An hour later the body temperature has decreased to 60 °. It's been a winter inversion in SLC, with constant ambient temperature 30 °. Assuming the Newton's law model, estimate the time of death. Hint: Begin by finding formulas for the functions $T(t)$ that solve this "separable" differential equation.

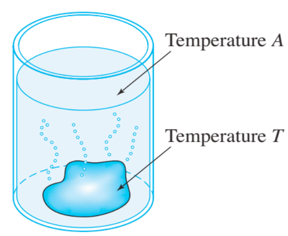


FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

1) Solve $\frac{dT}{dt} = -k(T - A)$

separable $\int \frac{dT}{T-A} = \int -k\, dt$

$$\ln|T-A| + C_1 = -kt + C_2$$

$$e^{\ln|T-A|} = e^{-kt + C_3}$$

$$|T-A| = e^{-kt} \underbrace{e^{C_3}}_{>0}$$

$$T - A = \underbrace{\pm e^{C_3}}_{C} e^{-kt}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

3 unknowns A, C, k; $A = 30°$

3 pieces of data to determine them

...

Wed

$t = 0$ be 3:00 pm.
measure time in hours

$$T(t) = 30 + Ce^{-kt}$$

$$T(0) = 70 = 30 + C$$

$$T(1) = 60 = 30 + Ce^{-k}$$

So $C = 40$.

$$T(1) = 60 = 30 + 40e^{-k}$$

$$30 = 40e^{-k}$$

$$.75 = e^{-k}$$

$$\left(\ln(.75) = -k \quad k = -\ln(.75) \right)$$

So

$$T(t) = 30 + 40(e^{-k})^t$$

$$\boxed{T(t) = 30 + 40(.75)^t}$$

Set $T = 98.6$, solve for t

$$98.6 = 30 + 40(.75)^t$$

$$\frac{68.6}{40} = \cancel{40}(.75)^t$$

$$\ln\left(\frac{68.6}{40}\right) = t \cancel{\ln(.75)}$$

$$\frac{\ln\left(\frac{68.6}{40}\right)}{\ln(.75)}$$

$$-1.875 = t$$

clock time $3 - 1.875 = 1.125$ O'clock

$\approx$ 1:075 O'clock.

Math 2280-1
Tues Jan 8
1.2, 1.4 antidifferentiation differential equations and separable differential equations

Announcements: HW you can try after what we discuss today includes

1.1 6, 15, 27, 29, 30, 34
1.2 w1.1 abc
2, 6, 7, 9
1.3 2, 6
w1.3
1.4 4, 12, 20

- Quiz tomorrow

I'll usually start class with a warm-up exercise (starting 5 minutes early)

'til 12:57

Warm-up Exercise: Use antidifferentiation to solve the initial value problem for y(x)

$$\text{IVP}\ \begin{cases} \frac{dy}{dx} = x\sqrt{x^2+4} \\ y(0) = 0 \end{cases}$$

soln $y(x) = \int x\sqrt{x^2+4}\,dx \quad = \frac{1}{3}(x^2+4)^{3/2} - \frac{8}{3}$ $\leftarrow +C$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$y(x) = \int u^{1/2}\tfrac{1}{2}du$$
$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C$$
$$y = \frac{1}{3}(x^2+4)^{3/2} + C$$
$$y(0) = 0 = \frac{1}{3}4^{3/2} + C$$
$$-\frac{1}{3}4^{3/2} = C$$
$$-8/3 = C$$

check! $y(0) = \frac{1}{3}4^{3/2} - \frac{8}{3} = \frac{1}{3}8 - \frac{8}{3} = 0$ ✓

$y'(x) = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(x^2+4)^{1/2}\cancel{2}x$
$= x\sqrt{x^2+4}$ ✓

Section 1.2 is about differential equations equivalent to ones of the form

$$\frac{dy}{dx} = f(x)$$

which we solve by direct antidifferentiation, as you learned in Calculus.

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

this was our warmup

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$

$$y(0) = 0$$

Section 1.4 is about *separable* differential equations which is a generalization that includes those of section 1.2:

Definition: A *separable* first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

Solution (chain-rule justified): One can rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0) .$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Taking antiderivatives with respect to the variable x we have

$$\int g(y)\frac{dy}{dx}\, dx = \int f(x)\, dx.$$

If $G(y)$ is any antiderivative of $g(y)$ with respect to the variable y then

$$g(y(x))\frac{dy}{dx} = G'(y(x))y'(x)$$

which by the chain rule (read backwards) is precisely

$$\frac{d}{dx}G(y(x)).$$

So we have

$$\int \frac{d}{dx}G(y(x))dx = \int f(x)\, dx ,$$

which we antidifferentiate with respect to x and obtain

$$G(y(x)) = F(x) + C.$$

where $F(x)$ is any particular antiderivative of $f(x)$. This identity

$$G(y) = F(x) + C$$

expresses solutions $y(x)$ *implicitly* as functions of x . (By differentiating this identity implicitly as you did in Calculus, you recover the original differential equation.)

You may be able to use algebra to solve this equation *explicitly* for $y = y(x)$ as we did for $T = T(t)$ in the Newton's Law of cooling problem.

Solution (differential magic for doing the computation quickly): Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx , and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx \, .$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx \text{ , i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C \, . \quad \text{Agrees!}$$

This differential magic is related to the "method of substitution" in antidifferentiation, which is essentially the "chain rule in reverse" for integration techniques.

Discuss on Wed.

<u>Exercise 2</u>: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

a) Use separation of variables to find solutions to this DE.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a?
c) Explain why the IVP

$$\left.\begin{aligned} \frac{dy}{dx} &= 1 + y^2 \\ y(0) &= 0 \end{aligned}\right\}$$

Wed warmup:

$$y(x) = \tan x = \frac{\sin x}{\cos x}$$

$\cos x = 0$ at $\pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \ldots$

has a solution, but this solution does not exist for all x.

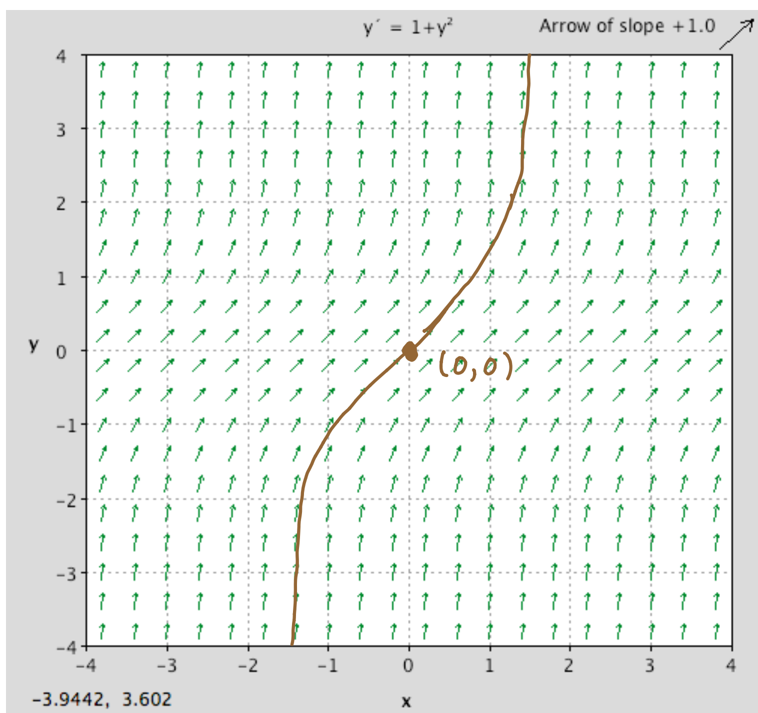
You can download the java applet "dfield" from the Rice University URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

(c) largest interval containing $x_0 = 0$ soln to IVP exists on

$-\pi/2 < x < \pi/2$

slope fcn at $(x,y) = 1 + y^2$

(b).



Section 1.2 applications:

An important class of antidifferentiation differential equations applications arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

$m x''(t) = \text{net forces}$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

$$v(t) = x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad \frac{m}{sec} \qquad a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} \quad \frac{m/s}{s}$$

$= x''(t)$

Exercise 3:

a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.)

b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

velocity: $\frac{m}{sec}$ or $\frac{ft}{sec}$. $\quad$ accel $\quad \frac{m}{s^2} \quad \frac{ft}{s^2}$

Exercise 4: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof. (Use the approximate value for the acceleration due to gravity, $g = 9.8 \ \frac{m}{s^2}$.)

a) Neglecting friction, how high will the object get above ground?

b) When does the object land ?

$y(5.102) \approx 157.6$ m

ans $\quad$ 10.2 sec $\quad$ (&-6.12)

Exercise 5:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a\,.$$

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a}\,.$$

this is in HW.

(This formula may help with some homework problems.)

c) Adapt the answer to b to check part of your work in Exercise 4.

5a) $x''(t) = -a \quad (a>0)$

$$x'(t) = \int x''(t)\,dt = \int -a\,dt = -at + C.$$

$$v(t) = x'(t) = -at + C$$

@ $t=0$ $\quad v(0) = 0 + C \Rightarrow C = v(0) = v_0$

$$\boxed{x'(t) = -at + v_0}$$

So $\quad x(t) = \int -at + v_0\,dt = -\frac{a}{2}t^2 + v_0 t + C$

@ $t=0$: $\quad x(0) = x_0 = 0 + 0 + C$

$$\boxed{x(t) = -\frac{a}{2}t^2 + v_0 t + x_0}$$

Math 2280-002
Wed Jan 9
1.3-1.4 more slope fields; existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Announcements:

- For today's quiz, you'll solve a separable DE.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

...

- Finish M, T notes today
  on Fri, we'll do Wed notes

'til 12:57

Warm-up Exercise: Use separation of variables to solve the initial value problem for y(x):

$$\begin{cases} \frac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$$

ans $\boxed{y = \tan x}$!

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\arctan(y) = x + C$$

tan: $y = \tan(x+C)$

@ $x = 0$: $0 = \tan(C)$

can choose $C = 0$.
since $\tan(0) = 0$

$$\left(\begin{array}{ll} & \sin^2 x + \cos^2 x = 1 \\ \div \cos^2 x: & \tan^2 x + 1 = \sec^2 x \\ & (y^2 + 1 = y') \end{array} \right)$$

On Friday :

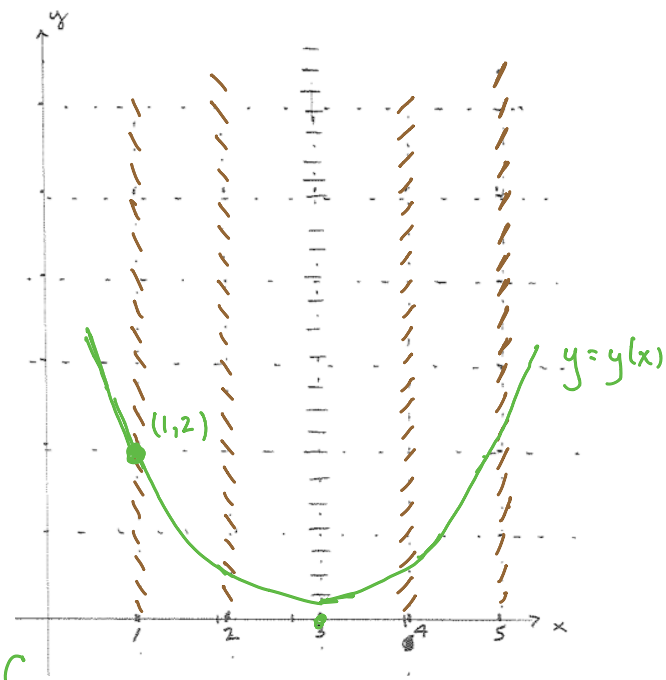
"slope fcn" only depends on x.

Exercise 1: Consider the differential equation $\boxed{\frac{dy}{dx} = x - 3}$, and then the IVP with $\boxed{y(1) = 2}$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \leq x \leq 5, 0 \leq y \leq 6$. Notice that in this example the value of the slope field only depends on *x*, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

a)

value of slope	slope fcn $x-3$
0	$x-3=0 \quad x=3$
1	$x=4$
-1	$x-3=-1 \Rightarrow x=2$
2	$x=5$
-2	$x-3=-2 \Rightarrow x=1$
	could fill in more!



c) $y'(x) = x-3$

$y(x) = \int x - 3 \, dx = \frac{1}{2}x^2 - 3x + C$

$y(1) = 2 = \frac{1}{2} - 3 + C$

$2 = -2.5 + C$

$4.5 = C$

$C = \frac{9}{2}$

$y(x) = \frac{1}{2}x^2 - 3x + \frac{9}{2}$

$= \frac{1}{2}(x^2 - 6x + 9)$

$y(x) = \frac{1}{2}(x-3)^2$

parabola with vertex (3,0)

The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

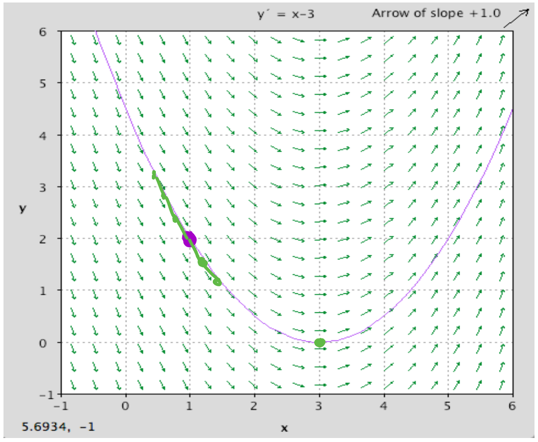
$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0), by picking a step size $\triangle x$ and then incrementing y by

$$\triangle y = f(x, y) \triangle x.$$

In this way one can *approximate* solution functions to initial value problems, and their graphs. The Java applet "dfield" (stands for "direction field", which is a synonym for slope field) uses (a more sophisticated analog of) this method to compute approximate solution graphs.

Here's a picture like the one we sketched by hand on the previous page, created by dfield.



Exercise 2: Consider the IVP

LHS RHS

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

NOT separable!! (but "linear" §1.5)

a) Check that $y(x) = x + 1 + C\,e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

c) Sketch the solution by hand, for the rectangle $-3 \leq x \leq 3, -3 \leq y \leq 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.

a) check $y(x) = x + 1 + Ce^x$ solves DE

LHS $y'(x) = 1 + Ce^x$

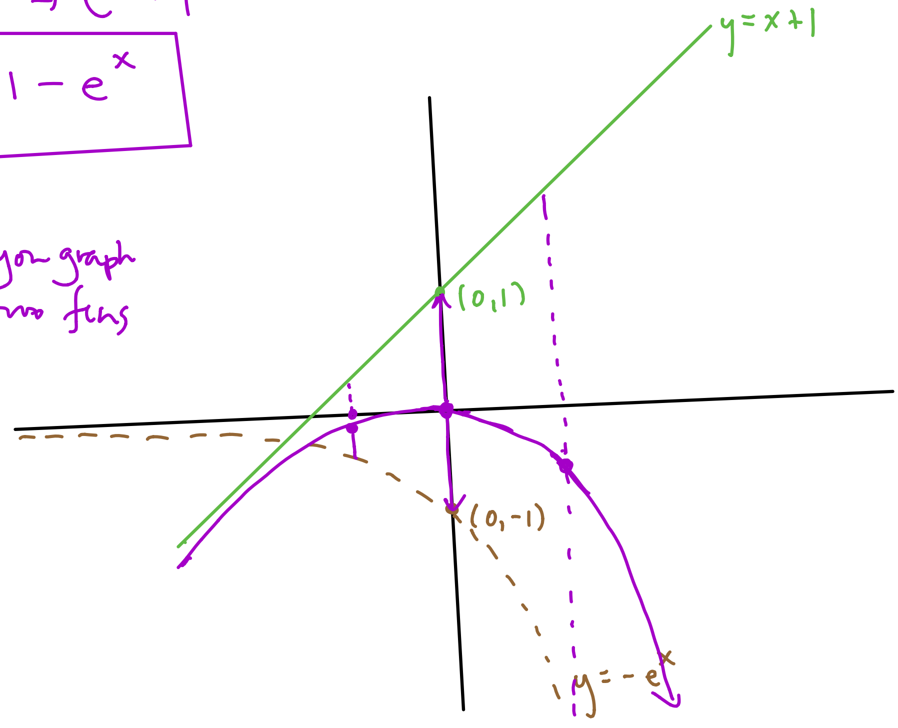
RHS $y - x = y(x) - x = \cancel{x} + 1 + Ce^x - \cancel{x} = 1 + Ce^x$

since LHS = RHS, $y(x)$ are solns

b) $y(0) = 0 = 1 + C \Rightarrow C = -1$

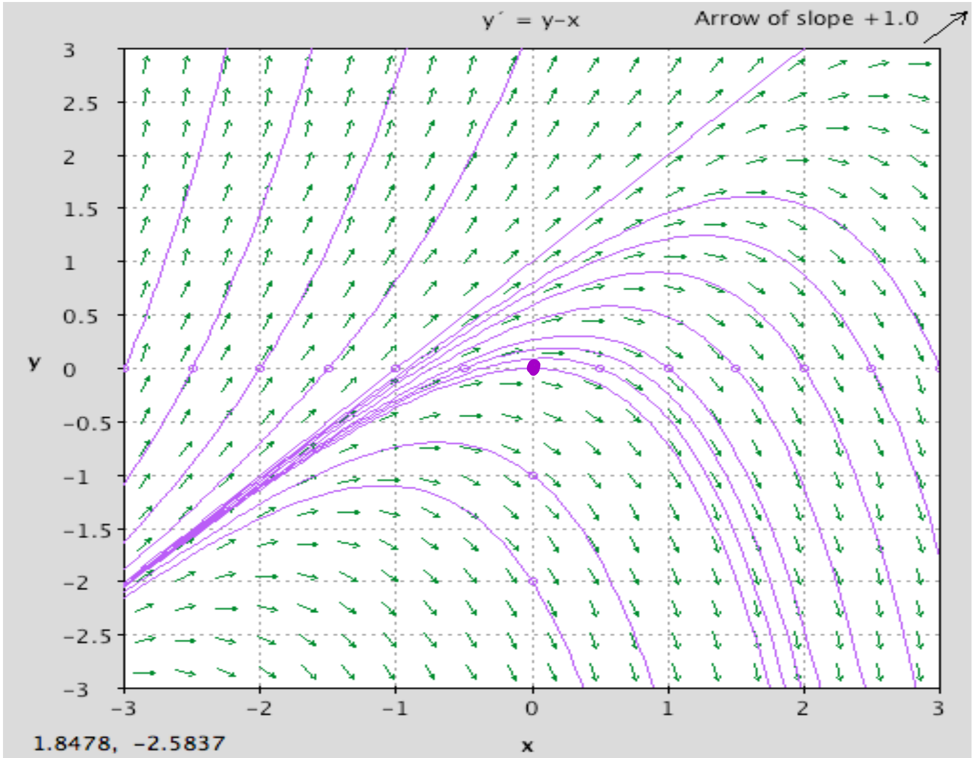
$$\boxed{y(x) = x + 1 - e^x}$$

review of how you graph the sum of two fcns



Moral of Ex 1, 2:

knowing slope field geometry is "same" as knowing geometry of the family of solution graphs



Exercise 3a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(0) = 0$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

3c) Sketch some of these singular solutions onto the slope field below.

$$\frac{dy}{dx} = f(x)\,g(y)$$
$$\frac{dy}{g(y)} = f(x)\,dx$$

Assuming $g(y) \neq 0$

If $g(y^*) = 0$ then $y(x) = y^*$ is a sol'n. (y^* const)

$y'(x) = 0$

$f(x)g(y) = f(x)g(y^*) = f(x)\cdot 0 = 0$

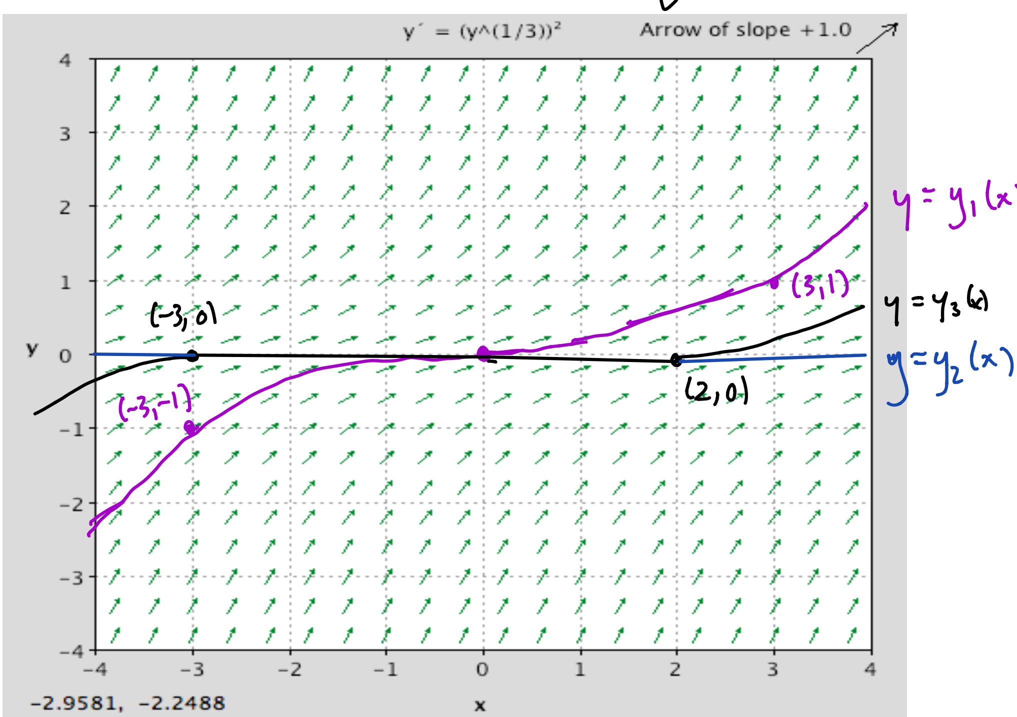
warmup 3a). We got sol's to the DE $y(x) = \frac{1}{27}(x+c)^3$

so for IVP $y_1(x) = \frac{1}{27}x^3$

3b) also noticed $y_2(x) \equiv 0$ solves IVP.

keyed into that via sep. of variables

∞'ly many sol'ns $\quad y_3(x) = \begin{cases} 0 & -3 \leq x \leq 2 \quad \text{"2" was arbitrary} \\ \frac{1}{27}(x-2)^3 & x \geq 2 \\ \frac{1}{27}(x+3)^3 & x < -3 \quad \text{"-3" was arbitrary} \end{cases}$

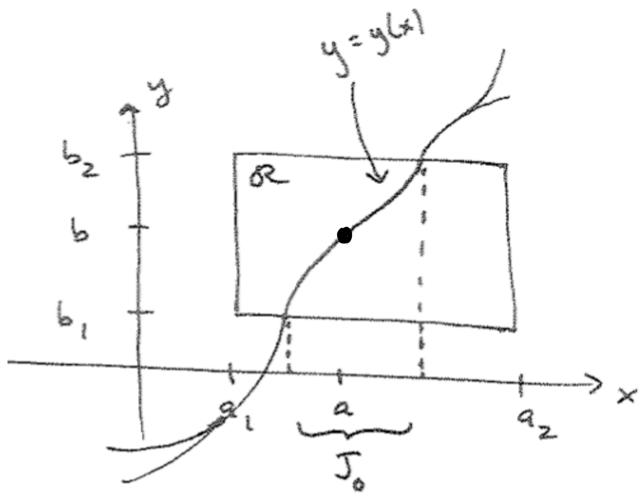


Here's what's going on (stated in 1.3 page 22 of text as *Theorem 1*; partly proven in Appendix A.)
Existence - uniqueness theorem for the initial value problem
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.
• Existence: If $f(x, y)$ is continuous in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
• Uniqueness: If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



Exercise 4: Discuss how the existence-uniqueness theorem is consistent with our work in Exercises 1-3 in today's notes.

Math 2280-001
Fri Jan 11
1.3-1.4 more slope fields and existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

<u>Announcements:</u>

- CANVAS is up, Chptrs 1.1-1.5 are in "files"
- Office Hours LCB 204
  M: 10:45 - 11:35 a.m.
  T: 2:00 - 3:00 p.m.
  also available briefly after class, and by appointment.
- today do Wed notes

'til 12:57

<u>Warm-up Exercise</u>: Use separation of variables to solve for $y(x)$

$$\begin{cases} \frac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$$

We'll use this today.

$$y^{-2/3}\,dy = 1\cdot dx$$

$$\left(\frac{1}{y^{2/3}}\,dy\right)$$

assumed $y \neq 0$

$$\int y^{-2/3}\,dy = \int 1\cdot dx$$

$$3y^{1/3} + C_1 = x + C_2$$

$$3y^{1/3} = x + C$$

$$y^{1/3} = \frac{1}{3}(x + C)$$

$$\boxed{y = \frac{1}{27}(x + C)^3}$$

if $y(0) = 0 \Rightarrow C = 0$
gave $y_1(x)$.

<u>a</u> solution
$y_1(x) = \frac{1}{27}x^3$.

$y_1(0) = 0$ ✓

LHS $y_1'(x) = \frac{1}{27}3x^2 = \frac{x^2}{9}$

RHS $\left(\frac{1}{27}x^3\right)^{2/3} = \frac{1}{9}x^2$ ✓

<u>Note</u> $y_2(x) \equiv 0$ is a solution to DE & IVP

$y_2(0) = 0$ ✓

LHS $y_2'(x) = 0$

RHS $(y_2(x))^{2/3} = (0)^{2/3} = 0$ ✓

OH OH.