Math 2280-002
Week 13, April 8-12, 5.7; 9.1-9.3
Matrix exponential summary and examples; Fourier series for periodic functions

Mon April 8
Matrix exponential summary, and discussion of computations for diagonalizable and non-diagonalizable matrices

Announcements:

Warm-up Exercise:

On Friday we computed e^{tA} using diagonalization, for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which had complex eigendata. (We left the last couple of multiplications because time ran out, but I filled those in for the posted notes.) Today we'll work an example which is more akin to your homework, with a diagonalizable matrix having real eigendata. Then we'll see an example of what can happen for non-diagonalizable matrices...you have a simple example in your homework assignment, and I'll also give an overview what happens in general, with most details omitted. The discussion is related to "Jordan Canonical Form" for matrices, which is usually included in the course *Math 5310-5320, Introduction to Modern Algebra.* (See also the Wikipedia page on matrix exponentials, although it's probably too abbreviated to be understandable with respect to Jordan Canonical form, without further digging.) Our text goes into some detail on computing e^{tA} for nondiagonalizable matrices as well, although we won't cover those details in this class.

First, recall/summarize from last week

(1) Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

$$e^{A} := I + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{k!}A^k + \ldots$$

$$\left(\Rightarrow e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^k}{k!}A^k + \ldots. \right)$$

(2) If A and B commute, i.e. $AB = BA$, then

$$e^{A} e^{B} = e^{B} e^{A}; \qquad e^{A+B} = e^{A} e^{B}$$

$$\left(\Rightarrow e^{t(A+B)} = e^{tA} e^{tB} \quad \text{and} \quad \left(e^{tA}\right)^{-1} = e^{-tA} \right)$$

(3)

$$\frac{d}{dt} e^{tA} = A\, e^{tA} \quad \left(= e^{tA} A \right).$$

In fact, e^{tA} is the unique $n \times n$ matrix $X(t)$ which satisfies

$$X'(t) = AX$$
$$X(0) = I\,.$$

($X(t) = e^{tA}$ solves this matrix IVP. If any other square matrix function also did, then each of its columns would solve the same IVP's as those of e^{tA}, so would have to be the columns of e^{tA} by uniqueness of solutions to IVP's.)

(4) If

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

then

$$e^{tD} = I + tD + \frac{t^2}{2!}D^2 + \dots + \frac{t^k}{k!}D^k + \dots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2!}\lambda_1^2 t^2 + \dots & 0 & \dots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}\lambda_2^2 t^2 + .. & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 + \lambda_n t + \frac{1}{2!}\lambda_n^2 t^2 + \dots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}.$$

(5) If A be diagonalizable and $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_n\}$ is an eigenbasis for $\mathbb{R}^n$ or $\mathbb{C}^n$ and if P is the invertible matrix with those eigenvectors as columns, and if D is the corresponding the diagonal matrix of eigenvalues so that

$$A\,P = P\,D$$
$$A = P\,D\,P^{-1}\ ,$$

Then

$$e^{tA} = I + tP\,D\,P^{-1} + \frac{t^2}{2!}\left(PDP^{-1}\right)^2 + \dots + \frac{t^k}{k!}\left(PDP^{-1}\right)^k + \dots$$
$$= P\left(I + tD + \frac{t^2}{2!}t^2 D^2 + \dots \frac{t^k}{k!}t^k D^k + \dots\right)P^{-1}$$
$$= P\,e^{tD}\,P^{-1}$$

(6) If the $n \times n$ matrix $\Phi(t)$ is a solution to

$$X'(t) = A X$$

(which is true if and only if each column is a solution to the homogeneous system $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$), and if the columns are a basis for the solution space to $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$, so that $\Phi(0)$ is invertible, then

$$X(t) = \Phi(t)\,\Phi(0)^{-1}$$

solves the square matrix IVP

$$\begin{aligned} X'(t) &= A X \\ X(0) &= I \end{aligned}$$

so

$$\Phi(t)\,\Phi(0)^{-1} = e^{tA}.$$

(see (3).)
In this case we call Wronskian $\Phi(t)$ a *Fundamental Matrix Solution (FMS)* for the linear system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}.$$

(7) In the case that A is diagonalizable, we can interpret (5,6) in terms of the Wronskian matrix whose columns are our usual basis for the homogeneous solution space:

$$e^{tA} = P\,e^{tD}\,P^{-1}$$

$$e^{tA} = \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} | & | & | & | \\ e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 & e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 & \ldots & e^{\lambda_n t}\underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \Phi(t)\,\Phi(0)^{-1}.$$

Exercise 1 For the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

the eigendata is

$$E_{\lambda=-2} = span\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\} \qquad E_{\lambda=5} = span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} .$$

Compute e^{tA}.

Exercise 2 Use the power series definition of matrix exponential to compute e^{tN} for

$$N = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

Definition A matrix N is called *nilpotent* if some power (and hence all higher powers) of the matrix is the zero matrix. (In fact, if N is an $n \times n$ nilpotent matrix, then it will always be true that $N^n = 0$, although a lower power of N might also be the zero matrix.)

Remark if N is nilpotent then the series for e^{tN} is finite and easy to compute, because after some point all the terms in the series are zero.

Exercise 3a   This ties in with Exercise 2, and deals with the only sort of non-diagonalizable case of a 2×2 matrix. Let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

$p(\lambda) = (\lambda + 2)^2$ but $E_{\lambda=-2}$ is defective:

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}.$$

Decompose A as

$$A = -2\,I + N = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix},$$

where N is the nilpotent matrix from the previous page. Multiples of the identity matrix commute with all other square matrices of the same size. Use this and your work in Exercise 2 to compute e^{tA}.

<u>Exercise 3b</u> The first order system with matrix A

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has solutions $[x(t), x'(t)]^T$ where $x(t)$ solves the critically damped DE

$$x'' + 4\,x' + 4\,x = 0.$$

Use this fact to re-compute e^{tA} from part a.

Although we only treated the 2×2 non-diagonalizable case, there is a theorem which generalizes what we did.....although the statement doesn't really tell you how to construct the decomposition for matrices which are larger than 2×2....this is also related to the Jordan Canonical form decomposition:

Theorem (Jordan–Chevalley decomposition) If A is not diagonalizable, then it is still possible to decompose A as

$$A = B + N$$

where B is diagonalizable, N is nilpotent, and $BN = NB$. Thus

$$e^{tA} = e^{tB + tN} = e^{tB} e^{tN}$$

is straightforward to compute once such a decomposition is known, as in Exercise 3a.

If you're curious, here's more of the story.

Recall from 2270:

<u>Theorem 1</u>: Let $A_{n\times n}$. Let $p(\lambda) = |A-\lambda I| = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$

$\lambda_1, \lambda_2, \ldots \lambda_\ell$ distinct

$k_1+k_2+\ldots+k_\ell = n$.

Then $1 \le \dim(E_{\lambda=\lambda_j}) \le k_j$

A is diagonalizable if and only if each $\dim(E_{\lambda=\lambda_j}) = k_j$.

i.e. in this case there is an invertible matrix P made out of eigenvector columns so that

$$AP = PD$$

$$D = \begin{bmatrix} \underbrace{\begin{matrix}\lambda_1 & 0 \\ 0 & \lambda_1\end{matrix}}_{k_1}\Big\}{\scriptstyle k_1} & & \bigcirc \\ & \underbrace{\lambda_2}_{k_2}\Big\}{\scriptstyle k_2} & \\ \bigcirc & & \underbrace{\begin{matrix}\lambda_\ell & 0 \\ 0 & \lambda_\ell\end{matrix}}_{k_\ell} \end{bmatrix}\Big\}{\scriptstyle k_\ell}$$

What we probably didn't teach you in 2270:

<u>Theorem 2</u> Let $A_{n\times n}$ as above, but with some defective eigenspaces, i.e.

$$\dim(E_{\lambda=\lambda_j}) < k_j$$

But then the larger "generalized eigenspace" defined as

$$G_{\lambda=\lambda_j} := Nul\,(A-\lambda_j I)^{k_j}$$

does satisfy

$$\dim(G_{\lambda=\lambda_j}) = k_j.$$

Generalized eigenvector bases can be chosen as the columns of an invertible matrix P so that

$$AP = PJ$$

where J is the "Jordan Canonical Form" of A, made of "Jordan blocks" along the diagonal, and zero elsewhere. Each Jordan block has an eigenvalue along the diagonal, ones along the "superdiagonal" and zeroes elsewhere:

1×1 block: $[\lambda]$, $\quad$ 2×2 block: $\begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}$ $\quad$ 3×3 block: $\begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}$ $\quad$ 4×4: $\begin{bmatrix}\lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda\end{bmatrix}$

etc.

Example Let $A_{4\times 4}$, $|A-\lambda I| = (\lambda-3)(\lambda-2)^3$. The Jordan Canonical form of A (up to ordering of the blocks) is one of:

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\dim E_{\lambda=2} = 3$

$\dim E_{\lambda=3} = 1$

A is diagonalizable

$$J = \left[\begin{array}{c|cc|c} 2 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$$

$\dim E_{\lambda=2} = 2$ (= # of $\lambda=2$ blocks)

$\dim E_{\lambda=3} = 1$

A not diagonalizable

$$J = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$$

$\dim E_{\lambda=2} = 1$

$\dim E_{\lambda=3} = 1$

A not diagonalizable

Every Jordan matrix can be written as

$$J = D + N$$

where D is the diagonal part and N is the superdiagonal part and is nilpotent; and, D commutes with N, $DN = ND$.

$$AP = PJ$$

$$A = PJP^{-1}$$

$$e^{tA} = Pe^{tJ}P^{-1}$$

$$= P \underbrace{e^{tD}}_{\substack{\text{diagonal} \\ \text{matrix with} \\ \text{entries } e^{t\lambda}}} \underbrace{e^{tN}}_{\substack{\text{finite} \\ \text{power series.}}} P^{-1}$$

Example For $J = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$, $D = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$ $N = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$

$$N^2 = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$N^3 = [0]$$

$$e^{tJ} = e^{tD}e^{tN}$$

$$= \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{3t} \end{bmatrix} \underbrace{\left[\begin{array}{ccc|c} 1 & t & t^2/2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{I + tN + \frac{t^2}{2!}N^2}$$

Tues April 9
9.1 Introduction to Fourier series

Announcements:

Warm-up Exercise:

Dot products and inner products

Flow chart of dot product geometry in $\mathbb{R}^n$:

$$\vec{x}\cdot\vec{y} := \sum_{j=1}^{n} x_j\cdot y_j$$

$\downarrow$ algebra

a) $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ (symmetry)

b) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$
$\vec{x}\cdot(c\vec{y}) = c\,\vec{x}\cdot\vec{y}$ (linear in each argument)

c) $\vec{x}\cdot\vec{x} \geq 0$; $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x}=\vec{0}$ (positive)

An <u>inner product space</u> is a (real scalar) vector space V together with an "inner product" $\langle\cdot,\cdot\rangle$ which gives a real number for each pair of vectors, so that the following axioms hold for all $f, g, h \in V$, $c \in \mathbb{R}$

a) $\langle f,g\rangle = \langle g,f\rangle$

b) $\langle f,g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$
$\langle f,cg\rangle = c\langle f,g\rangle$

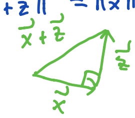
c) $\langle f,f\rangle \geq 0$. $\langle f,f\rangle = 0$ iff $f=0$.

from algebra... geometry

magnitude (norm)
$\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$

distance from $\vec{x}$ to $\vec{y}$ is $\|\vec{x}-\vec{y}\|$

orthogonality
$\vec{x}\perp\vec{z}$ iff $\vec{x}\cdot\vec{z}=0$
(iff Pythagorean Thm $\|\vec{x}+\vec{z}\|^2 = \|\vec{x}\|^2+\|\vec{z}\|^2$)

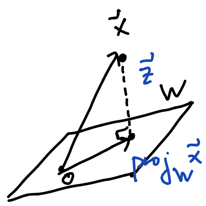


$$\|\vec{x}+\vec{z}\|^2 = (\vec{x}+\vec{z})\cdot(\vec{x}+\vec{z}) = \|\vec{x}\|^2 + \|\vec{z}\|^2 + 2\,\vec{x}\cdot\vec{z}.$$

From these algebra axioms the entire concept chart on the left also holds, for finite dimensional subspaces W.

orthogonal and orthonormal basis for subspaces W

Gram-Schmidt procedure for constructing orthogonal or ortho-normal bases for subspaces



$$\mathrm{proj}_W\,\vec{x} = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \dots + \frac{\vec{x}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k$$

is the <u>nearest point to $\vec{x}$</u> in W if $\{\vec{v}_1,\vec{v}_2,\dots\vec{v}_k\}$ is orthogonal basis for W

formula follows by solving for weights $c_1, c_2, \cdots c_k$ so that

$$\vec{z} := \vec{x} - (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k)$$

is $\perp$ to each $\vec{v}_j$

want

$$0 = \vec{z}\cdot\vec{v}_j = \vec{x}\cdot\vec{v}_j - c_j\,\vec{v}_j\cdot\vec{v}_j \Rightarrow c_j = \frac{\vec{x}\cdot\vec{v}_j}{\vec{v}_j\cdot\vec{v}_j}$$

Examples of function space inner products:

$$V = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous}\} := C([a, b]).$$

$$\langle f, g \rangle := \int_a^b f(t)\, g(t)\, \mathrm{d}t \qquad \text{(or some fixed positive multiple of this integral).}$$

<u>Exercise 1</u>) Check the algebra requirements a), b), c) for an inner product.

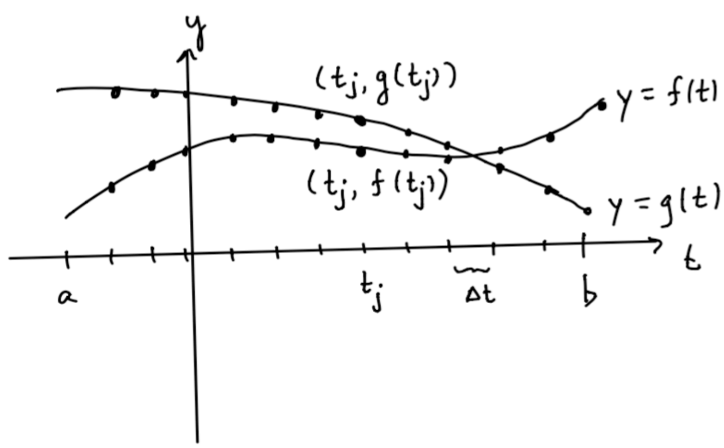
This inner product $\langle f, g \rangle$ is not so different from the $\mathbb{R}^n$ dot product if you think of Riemann sums: Let

$$\Delta t = \frac{b-a}{n}; \qquad t_j = a + j\,\Delta t, j = 1, 2, .. n.$$

Then

$$\langle f, g \rangle = \int_a^b f(t) g(t)\, \mathrm{d}t = \lim_{n \to \infty} \sum_{j=1}^{n} f(t_j)\, g(t_j) \Delta t$$

$$= \lim_{n \to \infty} \left(\begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix} \Delta t \right).$$



Fourier series

Let $f\colon [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuous function, or equivalently, extend to $f\colon \mathbb{R} \to \mathbb{R}$ as a 2π-periodic, piecewise continuous function.

Then the *Fourier coefficents of f* are computed via the definitions

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \mathrm{d}t$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\, \mathrm{d}t,\ n \in \mathbb{N}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\, \mathrm{d}t,\ n \in \mathbb{N}$$

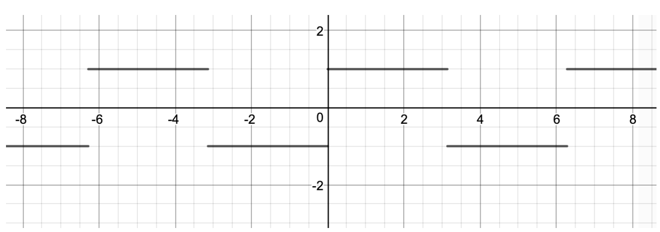
And the *Fourier series for f* is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

The idea is that the partial sums of the Fourier series of f should actually converge to f. The reasons why this should be true combine the linear algebra ideas related to orthogonal basis vectors and projection in the flow chart we just looked over, together with analysis ideas related to convergence. Let's do an example to illustrate the magic, before discussing (parts of) why the convergence actually happens.

Exercise 1 Let's consider the 2π-periodic "square wave" from your canceled homework problem:
The function $sq(t)$ is the 2π-periodic extension of

$$f(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$



Compute the Fourier coefficients and Fourier series for $sq(t)$.

Pictures of the graphs of the partial sums of the first "N" non-zero terms of the Fourier series, stolen from the text. We could've made these graphs with Desmos. Notice how the partial sums are doing their best to approximate the original 2π - periodic square wave we started with.

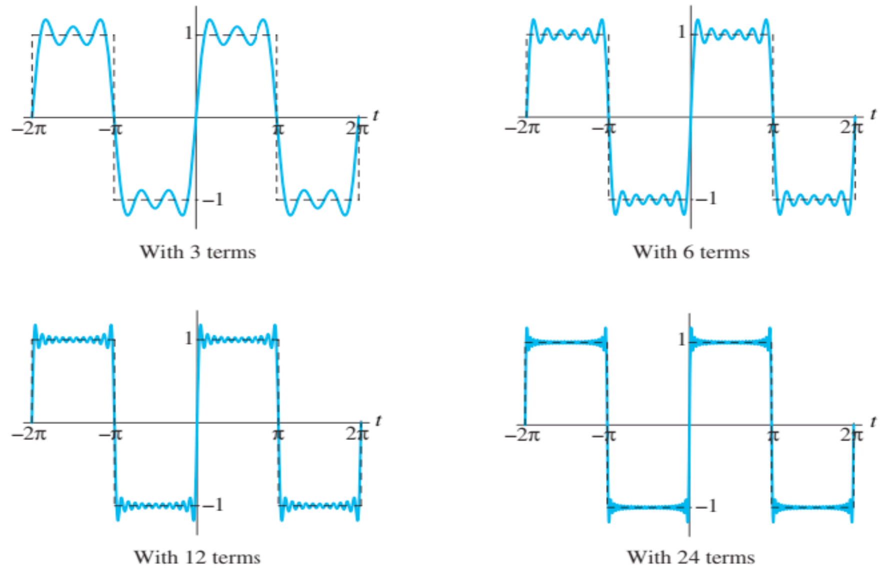


FIGURE 9.1.3. Graphs of partial sums of the Fourier series of the square-wave function (Example 1) with $N = 3, 6, 12,$ and 24 terms.

So what's going on?

<u>Theorem</u> Let $V=\{f:\mathbb{R}\to\mathbb{R}\ s.t.\ f$ is piecewise continuous and $2\,\pi$ - periodic$\}$. Define

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)g(t)\ \mathrm{d}t.$$

1) Then V, $\langle\ ,\ \rangle$ is an inner product space.
2) Let $V_N := span\{1, \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ... \sin(N\,t)\}$. Then the $2\,N+1$ functions listed in this collection are an orthogonal basis for the $(2\,N+1)$ dimensional subspace V_N. In particular, for any $f\in V$ the nearest function in V_N to f is given by the projection formula

$$proj_{V_N} f = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + \sum_{n=1}^{N} \frac{\langle f, \cos(nt)\rangle}{\langle \cos(nt), \cos(nt)\rangle}\cos(nt) + \sum_{n=1}^{N} \frac{\langle f, \sin(nt)\rangle}{\langle \sin(nt), \sin(nt)\rangle}\sin(nt)$$

and this works out to be precisely the truncated Fourier series

$$proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos(nt) + \sum_{n=1}^{N} b_n\sin(nt)$$

where a_0, a_n, b_n are the Fourier coefficients defined earlier:

$$\frac{a_0}{2} := \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f(t)\ \mathrm{d}t = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\ \mathrm{d}t = \frac{\langle f, \cos(nt)\rangle}{\langle \cos(nt), \cos(nt)\rangle}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\ \mathrm{d}t = \frac{\langle f, \sin(nt)\rangle}{\langle \sin(nt), \sin(nt)\rangle}$$

Exercise 2) Partially check that $\{1,\ \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ... \sin(N\,t)\}$ is orthogonal for our inner product, and also check why the Fourier coefficients match up to the inner product expressions.

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t$$

Hint:

$$\cos((m+k)t) = \cos(m\,t)\cos(k\,t) - \sin(m\,t)\sin(k\,t)$$
$$\sin((m+k)t) = \sin(m\,t)\cos(k\,t) + \cos(m\,t)\sin(k\,t)$$

so

$$\cos(m\,t)\cos(k\,t) = \frac{1}{2}(\cos((m+k)\,t) + \cos((m-k)t)) \quad \text{(even if } m = k\text{)}$$
$$\sin(m\,t)\sin(k\,t) = \frac{1}{2}(\cos((m-k)\,t) - \cos((m+k)t)) \quad \text{(even if } m = k\text{)}$$
$$\cos(m\,t)\sin(k\,t) = \frac{1}{2}(\sin((m+k)\,t) + \sin((-m+k)t)) \quad \text{(even if } m = k\text{)}$$

Convergence Theorems (These require some careful mathematical analysis to prove - they are often discussed in Math 5210, for example.)

Theorem 1 Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise continuous. Let

$$f_N = proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

be the Fourier series truncated at N. Then

$$\lim_{n \to \infty} \|f - f_N\| = \lim_{n \to \infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(t) - f_N(t)\right)^2 \mathrm{d}t \right]^{\frac{1}{2}} = 0.$$

In other words, the distance between f_N and f converges to zero, where we are using the distance function that we get from the inner product,

$$dist(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g\rangle} = \left(\int_{-\pi}^{\pi} (f(t) - g(t))^2 \,\mathrm{d}t \right)^{\frac{1}{2}}.$$

Theorem 2 If f is as in Theorem 1, and is (also) piecewise differentiable with at most jump discontinuities, then

(i) for any t_0 such that f is differentiable at t_0

$$\lim_{N \to \infty} f_N(t_0) = f(t_0) \quad \text{(pointwise convergence).}$$

(ii) for any t_0 where f is not differentiable (but is either continuous or has a jump discontinuity), then

$$\lim_{N \to \infty} f_N(t_0) = \frac{1}{2}\left(f_-(t_0) + f_+(t_0)\right)$$

where

$$f_-(t_0) = \lim_{t \to t_0^-} f(t), \quad f_+(t_0) = \lim_{t \to t_0^+} f(t)$$

Example: The truncated Fourier series for the square wave, i.e. the $sq_N(t)$, converge to $sq(t)$ for all t which are not multiples of π. At integer multiples of π the partial sums are all zero, and so is the limit. Zero is the average of the left and right hand limits of $sq(t)$ at these jump discontinuities.

Wed April 10

9.1-9.3 Differentiating and integrating Fourier series.

Announcements:

Warm-up Exercise:

Differentiating Fourier Series:

Theorem 3 Let f be 2π-periodic, piecewise differentiable and continuous, and with f' piecewise continuous. Let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then f' has the Fourier series you'd expect by differentiating term by term:

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} n\,b_n \cos(n\,t)$$

proof: Let f' have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t = \frac{1}{\pi} f(t)(-n)\sin(nt)\,]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(-n)\sin(n\,t)\,\mathrm{d}t$$

$$= 0 + \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = n\,b_n.$$

Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

Leads to

Integrating Fourier series:

Theorem 4 Let f be 2π-periodic piecewise continuous, and let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

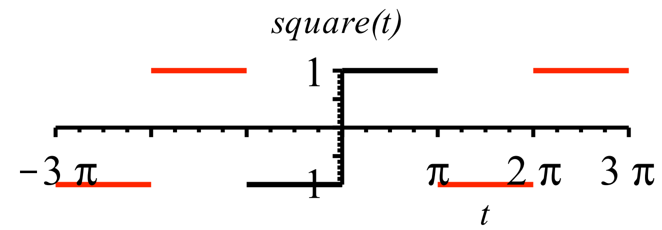
Then every antiderivative F of f is piecewise differentiable and can be found by integrating the Fourier series for f term by term:

$$F(t) = \frac{a_0}{2}t + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(n\,t) - \sum_{n=1}^{\infty} \frac{b_n}{n}\cos(n\,t) + C$$

(Note that $F(t)$ is only a periodic function if $a_0 = 0$.)

Exercise 1 On Tuesday we found the Fourier series for $sq(t)$, which is the $2\pi-$ periodic extension of

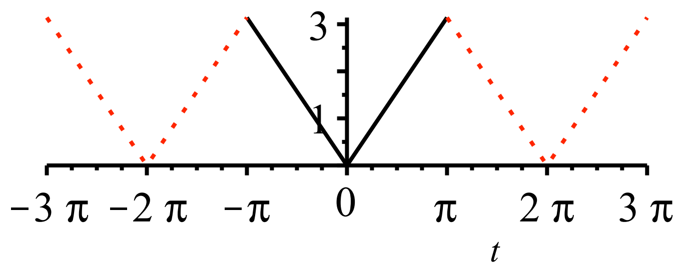
$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$



$$sq(t) = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin(n\,t).$$

Notice that the following "tent function", *tent(t)*, is an antiderivative of $sq(t)$. $tent(t)$ is the $2\pi-$periodic extension of $g(t) = |t|$ from the interval $[-\pi, \pi]$ to $\mathbb{R}$:

$$g(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$



Find the Fourier series for $tent(t)$ by antidifferentiation. Careful with the $\frac{a_0}{2}$ term! (There's a magic identity hiding in your formula once you've got it right.)

Exercise 2 For practice, find the Fourier series for $tent(t)$ by finding the Fourier coefficients directly from their definitions. You'll need to use integration by parts as well as facts about even and odd functions.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

$$\frac{a_0}{2} := \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}$$

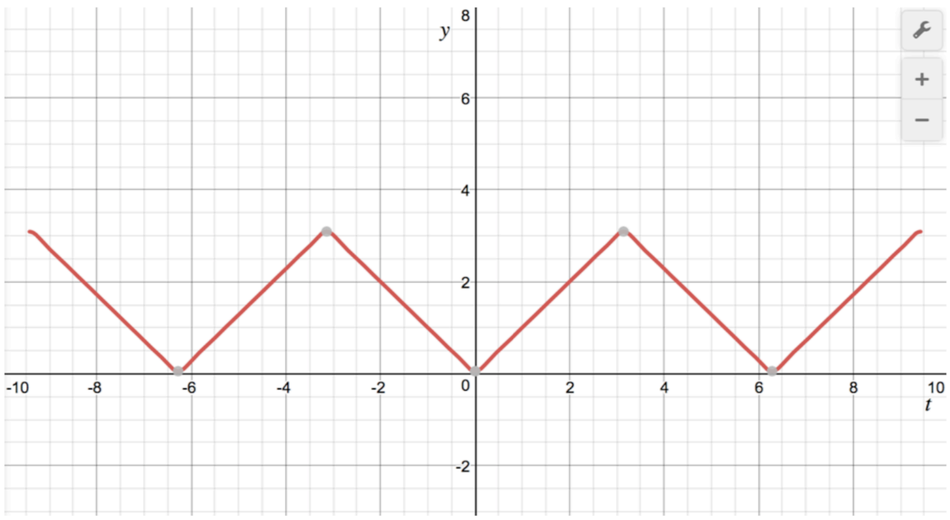
$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\,\mathrm{d}t = \frac{\langle f, \cos(nt)\rangle}{\langle \cos(nt), \cos(nt)\rangle}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = \frac{\langle f, \sin(nt)\rangle}{\langle \sin(nt), \sin(nt)\rangle}$$

At Desmos, this typed-in command:

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{5} \frac{1}{(2 \cdot j + 1)^2} \cdot \cos\left((2 \cdot j + 1) \cdot t\right) \left\{-3 \cdot \pi < t < 3 \cdot \pi\right\}$$

yielded this graph:



Fri April 12

9.2-9.3 Fourier series for $2L$ - periodic functions; cosine and sine series for functions defined on the interval $[0, L]$ and extended into either even or odd $2L$ - periodic functions.

Announcements:

Warm-up Exercise:

Fourier series for $2L$ - periodic functions:

So far we've only talked about Fourier series for 2π-periodic functions. In applications we want to be able to vary the period, and consider $2L$ - periodic functions instead, where L can be specified in the application. There's no problem in doing so:

<u>Theorem</u> Let $V = \{f : \mathbb{R} \to \mathbb{R} \ s.t. f$ is piecewise continuous and $2L -$ periodic$\}$.

Define the Fourier series for f by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right)$$

where the Fourier coefficients of f are defined analogously as for the 2π - periodic case. Note that the Fourier coefficients can again be expressed as projection weights with respect to an (adapted) inner product

$$\langle f, g \rangle := \int_{-L}^{L} f(t) g(t) \, dt.$$

$$\frac{a_0}{2} := \frac{1}{2L} \int_{-L}^{L} f(t) \, dt = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \text{the average value of } f.$$

$$a_n := \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L}t\right) dt = \frac{\left\langle f, \cos\left(\frac{n\pi}{L}t\right)\right\rangle}{\left\langle \cos\left(\frac{n\pi}{L}t\right), \cos\left(\frac{n\pi}{L}t\right)\right\rangle}$$

$$b_n := \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L}t\right) dt = \frac{\left\langle f, \sin\left(\frac{n\pi}{L}t\right)\right\rangle}{\left\langle \sin\left(\frac{n\pi}{L}t\right), \sin\left(\frac{n\pi}{L}t\right)\right\rangle}$$

So the truncated Fourier series is the projection of f onto the $2N + 1$ dimensional subspace

$$V_N := span\left\{1, \cos\left(\frac{\pi}{L}t\right), \cos\left(\frac{2\pi}{L}t\right), \ldots, \cos\left(\frac{N\pi}{L}t\right), \sin\left(\frac{\pi}{L}t\right), \sin\left(\frac{2\pi}{L}t\right), \ldots \sin\left(\frac{N\pi}{L}t\right)\right\}$$

$$proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}t\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi}{L}t\right).$$

The same convergence theorems, and integration/differentiation theorems hold as for the 2π - periodic case.

One reason the same theorems hold for the $2\,L-$ periodic functions and their Fourier series, as for the $2\,\pi-$periodic ones, is because it's possible to change the periods of functions by scaling the input variables, and relate the corresponding facts that way:

Let f, g be $2\,L-$periodic, with the inner product

$$\langle f, g\rangle := \int_{-L}^{L} f(t)g(t)\,\mathrm{d}t.$$

Change variables, letting

$$t = \frac{L}{\pi}\tau\,, \qquad dt = \frac{L}{\pi}\,d\tau$$

Then $-L \le t \le L$ corresponds to $-\pi \le \tau \le \pi$. In terms of the inner products,

$$\int_{-L}^{L} f(t)g(t)\,\mathrm{d}t = \frac{L}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\tau\right)g\left(\frac{L}{\pi}\tau\right)\,\mathrm{d}\tau$$

$$\frac{1}{L}\int_{-L}^{L} f(t)g(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\tau\right)g\left(\frac{L}{\pi}\tau\right)\,\mathrm{d}\tau.$$

In particular,

1) The $2\,L-$ periodic functions

$$\left\{1, \cos\left(\frac{\pi}{L}\,t\right), \cos\left(\frac{2\,\pi}{L}t\right), \ldots, \sin\left(\frac{\pi}{L}t\right), \sin\left(\frac{2\,\pi}{L}t\right), \ldots\right\}$$

correspond to the $2 - \pi$ periodic functions

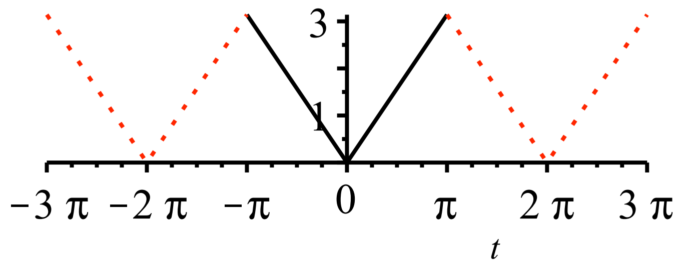
$$\{1, \cos(\tau), \cos(2\tau), \ldots, \sin(\tau), \sin(2\,\tau), \ldots\}$$

and the first collection is orthogonal with respect to the $2\,L-$ periodic function inner product because the second collection is orthogonal with respect to the $2\,\pi-$periodic function inner product.

2) If $f(t)$ is $2\,L-$periodic, then its Fourier coefficients are the same as those for the $2\,\pi-$periodic function $f\left(\frac{L}{\pi}\tau\right)$; If $g(\tau)$ is $2\,\pi-$periodic, then its Fourier coefficients are the same as those for the $2\,L-$periodic function $g\left(\frac{\pi}{L}t\right)$.

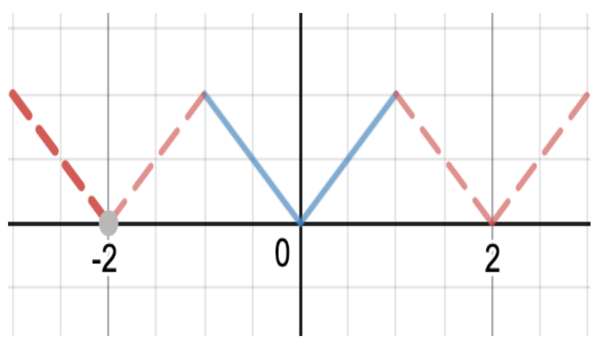
Exercise 1 Use the Fourier series for $2\pi-$ tent function to find the Fourier series for a tent function with period 2. Careful! (But if you do it right you save a lot of time over recomputing all of the Fourier coefficients using the formulas for $2L-$periodic functions!)

$$tent(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$



$$tent(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos(n\,t)$$

$$f(t) = \begin{cases} -t & -1 < t < 0 \\ t & 0 < t < 1 \end{cases}$$



Even and odd extensions of functions defined on $[0, L]$, into functions of period $2L$:

Let $f(t)$ be defined on the interval $[0, L]$. You can extend it into an even function on the interval $[-L, L]$ by

$$f_{even}(t) = \begin{cases} f(t) & 0 < t < L \\ f(-t) & -L < t < 0 \end{cases}$$

and then you can extend $f_{even}(t)$ to be a $2L-$ periodic even function. The Fourier coefficients for $f_{even}(t)$ are given by

$$\frac{a_0}{2} := \frac{1}{2L}\int_{-L}^{L} f_{even}(t)\,\mathrm{d}t = \frac{1}{L}\int_{0}^{L} f(t)\,\mathrm{d}t = \text{the average value of } f.$$

$$a_n := \frac{1}{L}\int_{-L}^{L} f_{even}(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t \quad \text{(even function integrand)}$$

$$b_n := \frac{1}{L}\int_{-L}^{L} f_{even}(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t = 0 \quad \text{(odd function integrand)}$$

So $f_{even}(t)$ is expressed with a cosine series

$$f_{even} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right)$$

If you take the same function f you can alternately extend it as an odd function on the interval $[-L, L]$ by

$$f_{odd}(t) = \begin{cases} f(t) & 0 < t < L \\ -f(-t) & -L < t < 0 \end{cases}$$

and then you can extend $f_{odd}(t)$ to be a $2L-$ periodic odd function. The Fourier coefficients for $f_{odd}(t)$ are given by

$$\frac{a_0}{2} := \frac{1}{2L}\int_{-L}^{L} f_{odd}(t)\,\mathrm{d}t = 0 = \text{the average value of } f_{odd}.$$

$$a_n := \frac{1}{L}\int_{-L}^{L} f_{odd}(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t = 0 \quad \text{(odd function integrand)}$$

$$b_n := \frac{1}{L}\int_{-L}^{L} f_{odd}(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t \quad \text{(even function integrand)}$$

So $f_{odd}(t)$ is expressed with a sine series

$$f_{odd} \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right).$$

The amazing fact is that both the even and odd extension Fourier series of a function original defined only on $[0, L]$ converge to the original function on that interval (in the same ways that convergence holds for general Fourier series). We will use these even and odd extensions when we move to the study of partial differential equations.

For example, the even periodic extension of $f(t) = t, 0 < t < \pi$ is the tent function, whereas the odd extension is the sawtooth function. I've superposed truncated Fourier series graphs for each, created at Demos: