Math 2280-002
Week 12, April 1-4, 5.4, 5.6-5.7
Continuing discussion of unforced and forced mass-spring systems; matrix exponentials and applications.

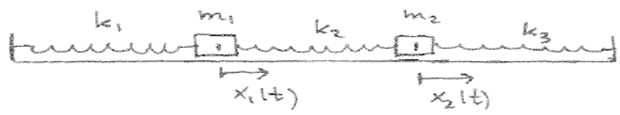
Mon April 1

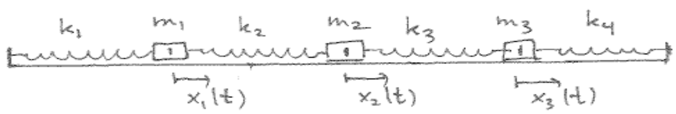
5.4 mass-spring systems, and forced oscillations

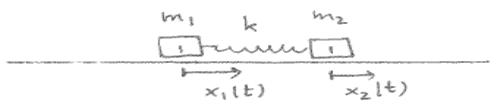
Announcements:

Warm-up Exercise:

Before the exam we were studying unforced mass-spring systems ...







Newton's law leads to a second order system of differential equations for the vector of displacements of the masses from their equilibrium locations,

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}.$$

If there are n masses the solution space is $2\,n$- dimensional, because for each mass you can specify initial displacement and velocity in the IVP.

<u>Solution space algorithm:</u> Consider the homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

<u>If</u> $A_{n \times n}$ is a diagonalizable matrix and <u>if all of its eigenvalues are non-positive</u> then for each <u>eigenpair</u> $(\lambda_j, \underline{\boldsymbol{v}}_j)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos(\omega_j\, t)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin(\omega_j\, t)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

And for an eigenpair $(\lambda_j, \underline{\boldsymbol{v}}_j)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions

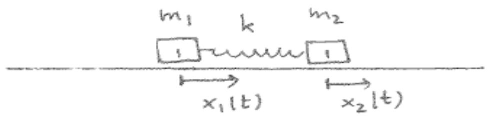
$$\boldsymbol{x}_j(t) = \underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = t\,\underline{\boldsymbol{v}}_j$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

<u>Remark</u>: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall, all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution solutions $(c_1 + c_2\, t)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

We had analyzed a two-mass, three spring configuration in which the masses and springs were all the same, and conducted an experiment which tested the model. We were in the middle of this exercise, although last Wednesday we set $m_1 = m_2$ for simplicity ...

Exercise 1) Consider a train with two cars connected by a spring:



1a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1{}'' = \frac{k}{m_1}(x_2 - x_1)$$

$$x_2{}'' = -\frac{k}{m_2}(x_2 - x_1)$$

1b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ remember that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in your homework problem about CO_2 vibrations.

Input:

eigenvalues	$\begin{pmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{pmatrix}$

Results:

$\lambda_1 = 0$

$\lambda_2 = \frac{k(-m_1 - m_2)}{m_1 m_2}$

Corresponding eigenvectors:

$v_1 = (1, 1)$

$v_2 = \left(-\frac{m_2}{m_1}, 1\right)$

Forced oscillations (still undamped):

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\boldsymbol{x}}''(t) = A\underline{\mathbf{x}} + M^{-1}\underline{\boldsymbol{F}}(t) .$$

If the forcing is sinusoidal,

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{G}_0}$$
$$\Rightarrow \quad \underline{\boldsymbol{x}}''(t) = A\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{F}_0}$$

with $\underline{\boldsymbol{F}_0} = M^{-1}\underline{\boldsymbol{G}_0}$.

From vector space theory we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_P(t) + \underline{\mathbf{x}}_H(t) ,$$

Forced oscillation particular solution algorithm:

$$\underline{\boldsymbol{x}}''(t) = A\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{F}_0}$$

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega t)\,\underline{\boldsymbol{d}}$$

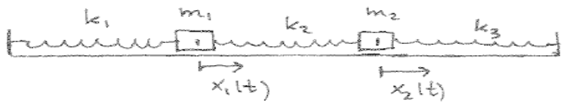
where the constant vector $\underline{\boldsymbol{d}}$ is what we need to find. (It's value will depend on the angular frequency ω of the forcing function.)

Exercise 2) Substitute the guess $\underline{\boldsymbol{x}}_P(t) = \cos(\omega t)\,\underline{\boldsymbol{d}}$ into the DE system

$$\underline{\boldsymbol{x}}''(t) = A\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{F}_0}$$

to find a matrix algebra formula for $\underline{\boldsymbol{d}} = \underline{\boldsymbol{d}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution: $\underline{\boldsymbol{d}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}_0}$. Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue, i.e. ω is not one of the natural frequencies.



Last week experimental configuration model:

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 3) Continuing with the configuration shown above, but now for a forced oscillation problem, let $k = m$ (one can do this by changing units of time to make the discussion completely general and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = \sqrt{\frac{k}{m}} = 1$, $\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{3}$ and that

$$\underline{\boldsymbol{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\boldsymbol{x}}_P(t)$, using the undetermined coefficients formula

$$\underline{\boldsymbol{d}}(\omega) = -\left(A + \omega^2 I\right)^{-1} \underline{\boldsymbol{F}}_0$$

Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix}$$

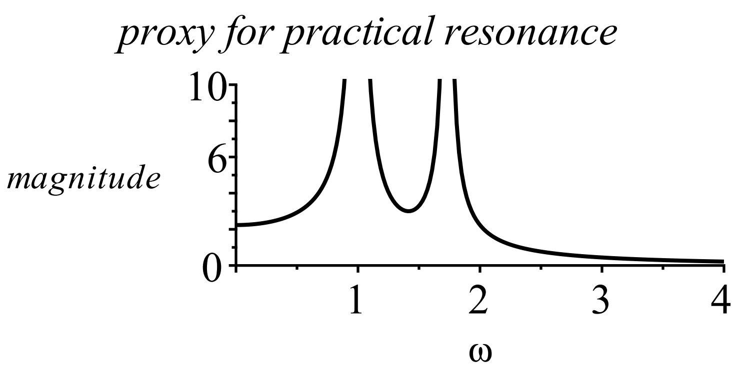
$$+ C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{\boldsymbol{d}}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{\boldsymbol{c}}(\omega)$ term for the undamped problem....see next page for visualizations.

```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> d := ω → (A + ω^2 · Iden)^(-1) . (-F0) : # the formula we worked out by hand
> d(ω);
```

$$\begin{bmatrix} \frac{3}{\omega^4 - 4\,\omega^2 + 3} \\ -\frac{3\left(\omega^2 - 2\right)}{\omega^4 - 4\,\omega^2 + 3} \end{bmatrix} \qquad \textbf{(1)}$$

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(d(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title = `proxy for practical resonance`);
      # Norm(c(ω),2) is the magnitude of the c(ω) vector
```

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>
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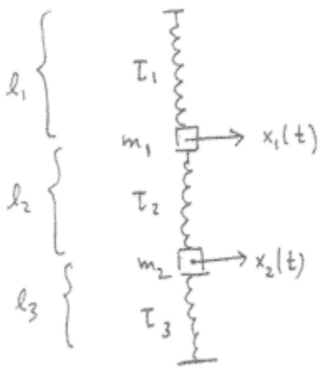
There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings...this is like one of your homework problems from a few weeks ago...

- Transverse oscillations! (i.e. directions ⊥ to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2+K_3)x_2$$

where K_1, K_2, K_3 are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{l_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{l_1} = -\frac{T_1}{l_1} x_1$$

So $K_1 = \frac{T_1}{l_1}$

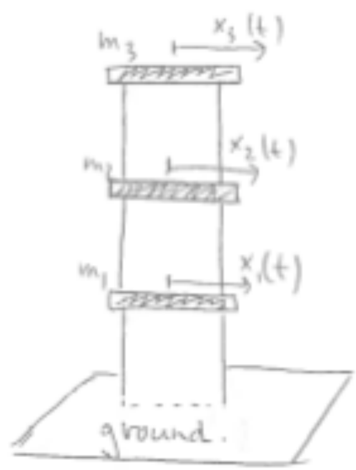
similarly, $K_2 = \frac{T_2}{l_2}$, $K_3 = \frac{T_3}{l_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k\,l$; $\frac{T}{l} \approx k$, so actually almost recover same fundamental modes !!]

• An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Exercise 3 Here is eigendata for the unscaled matrix $\left(\frac{k}{m} = 1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$ but the eigenvectors describing the modes would stay the same. Use this information describe the fundamental modes, and the order in which they will appear.

Input:

eigenvalues $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Results:

$\lambda_1 \approx -3.24698$

$\lambda_2 \approx -1.55496$

$\lambda_3 \approx -0.198062$

Corresponding eigenvectors:

$v_1 \approx (1.80194, -2.24698, 1)$

$v_2 \approx (-1.24698, -0.554958, 1)$

$v_3 \approx (0.445042, 0.801938, 1)$

Input:

eigenvalues $\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

Results:

$\lambda_1 \approx -2.61803$

$\lambda_2 \approx -0.381966$

Corresponding eigenvectors:

$v_1 \approx (-1.61803, 1)$

$v_2 \approx (0.618034, 1)$

Tues April 2
5.6 Matrix exponentials.

Announcements:

Warm-up Exercise:

Matrix exponentials. If you want to get a sense of the breadth of their applications in pure and applied math, consult the Wikipedia page on this topic! It also has a lot of the basic facts that we'll go through and use...

In the next three classes we'll talk about how matrix exponentials e^{tA} can be used to solve *all* homogeneous and non-homogeneous first order systems of differential equations with constant coefficient matrices A.

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

So I sort of lied when I told you earlier in the course that for higher order linear DE's and for first order systems of DE's, there weren't explicit formulas for the solutions. There *are* explicit formulas, as long as the coefficient matrix is constant. The formulas and method will look exactly like a matrix-vectorized version of the method for scalar first order linear differential equation solutions that we studied in Chapter 1, that we solve with exponential integrating factor - namely solutions to

$$x'(t) - a\,x = f(t).$$

Definitions and properties:

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

$$e^{A} := I + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{n!}A^n + \ldots$$

$$\left(\Rightarrow e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots. \right)$$

(1) Note: the infinite sum converges: Let M be the maximum of all of the absolute values of the entries a_{ij}. Then the maximum absolute value of any entry in A^2 is at most $M^2 + M^2 + \ldots + M^2 = nM^2$. So the maximum absolute value of entry in A^3 is at most n^2M^3, etc; the maximum absolute value of any entry of A^m is at most $n^{m-1}M^m$.

$$\left|entry_{ij}\, e^{A}\right| \le 1 + M + \frac{1}{2!}nM^2 + \frac{1}{3!}n^2M^3 + \ldots$$

$$\le 1 + nM + \frac{1}{2!}(nM)^2 + \ldots + \frac{1}{n!}(nM)^n + \ldots = e^{nM} < \infty.$$

Since the series for each entry is absolutely convergent, it is also convergent. So the entries of the limit matrix exist and are numbers with absolute value less than e^{nM}.

(2) If A and B commute, i.e. $AB = BA$, then

$$e^{A}e^{B} = e^{B}e^{A}.$$

(3) If A and B commute, then

$$e^{A+B} = e^A e^B \qquad \left(= e^B e^A \right)$$

(4) So

$$e^A e^{-A} = e^{A-A} = e^{[0]} = I.$$

In other words, e^A is always invertible, and its inverse matrix is e^{-A}.

Exercise 1 Use the power series definition to compute e^{tA} for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Theorem 1

a)

$$\frac{d}{dt}\, e^{tA} = A\, e^{tA}.$$

b) The j^{th} column of e^{tA} is the solution to the IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{e}}_j$$

where $\underline{\boldsymbol{e}}_j$ is the standard basis vector which is zero in each entry except for the j^{th} entry, which is 1.

c) The solution to the general homogenous IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

is

$$\underline{\boldsymbol{x}}(t) = e^{tA}\underline{\boldsymbol{x}}_0.$$

Compare to Chapter 1 for the scalar version!

Exercise 2 Verify Theorem 1abc for our matrix A in Exercise 1:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Remark: Theorem 1b indicates another way to compute e^{tA}: Its j^{th} column is the unique solution to the IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{e}}_j$$

We could have used that method to come up with e^{tA} in Exercise 2, especially since the first order system in this case is corresponds to the second order harmonic oscillator differential equation

$$x''(t) + x(t) = 0$$

There's a different way to compute matrix exponentials for diagonalizable matrices - which we'll discuss tomorrow.

Wed April 3
5.6-5.7   Matrix exponentials and fundamental matrix solutions continued.

Announcements:

Warm-up Exercise:

Yesterday we used power series to define matrix exponentials, and saw what e^{tA} has to do with solutions to systems of differential equations and corresponding IVP's

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

If the matrix A is diagonalizable there's a good method to compute e^{tA}, as the next two theorems indicate

Theorem 2 If

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

is a diagonal matrix, then

$$e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

Theorem 3 Let A be diagonalizable, i.e.

there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A. Let P be the invertible matrix with those eigenvectors as columns, and let D be the diagonal matrix which has the corresponding eigenvalues in the diagonal entries, i.e.

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n \right],$$

and

$$A\,P = P\,\mathrm{D}$$
$$A = P\,\mathrm{D}\,P^{-1}$$

Then

$$e^{t\,A} = P\,e^{t\,\mathrm{D}}\,P^{-1}$$

Remark and definition: Group the product expression for e^{tA} as follows:

$$e^{tA} = P\, e^{tD} P^{-1}$$

$$e^{tA} = \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} | & | & | & | \\ e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 & e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 & \cdots & e^{\lambda_n t}\underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \Phi(t)\, \Phi(0)^{-1}.$$

In this case we call $\Phi(t)$ a *Fundamental Matrix Solution (FMS)* for the linear system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\, \underline{\boldsymbol{x}}.$$

This is because every solution to the system can be written uniquely as a linear combination of the columns of $\Phi(t)$, i.e. as $\Phi(t)\underline{\boldsymbol{c}}$. We use the same definition in the case that A is not diagonalizable, namely that the columns of $\Phi(t)$ should be a basis for the solution space. And then it will always be true that

$$e^{tA} = \Phi(t)\, \Phi(0)^{-1}.$$

Exercise 1 Use Theorem 3 to recompute e^{tA} for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(This would be even easier if the eigendata was real instead of complex.)

Fri April 5
5.7 Matrix exponentials as integrating factors for nonhomogeneous systems of linear differential equations.

Announcements:

Warm-up Exercise:

Generalizing what we did in Chapter 1 for certain scalar first order linear DE's with constant coefficient, to systems, using integrating factors!

Formula for the solution to the IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

where A is a constant matrix. This is sort of amazing! We'll check every step:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

$$\mathrm{e}^{-t\,A}(\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}) = \mathrm{e}^{-t\,A}\underline{\boldsymbol{f}}(t)$$

$$\frac{d}{dt}\left(\mathrm{e}^{-t\,A}\underline{\boldsymbol{x}}(t)\right) = \mathrm{e}^{-t\,A}\underline{\boldsymbol{f}}(t)\ .$$

Integrate from 0 to t:

$$\mathrm{e}^{-t\,A}\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{x}}_0 = \int_0^t \mathrm{e}^{-s\,A}\underline{\boldsymbol{f}}(s)\ \mathrm{d}s$$

Move the $\underline{\boldsymbol{x}}_0$ over and multiply both sides by $\mathrm{e}^{t\,A}$:

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{t\,A}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-s\,A}\underline{\boldsymbol{f}}(s)\ \mathrm{d}s\right)\ .$$

This formula can be reorganized a bit, in a way which nicely isolates the homogeneous solution contribution related to the initial data $\underline{\boldsymbol{x}}_0$, added to the particular solution which is zero initial data:

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{t\,A}\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{(t-s)\,A}\underline{\boldsymbol{f}}(s)\ \mathrm{d}s\ .$$

This formula also illustrates a deep fact about linear differential equations and certain partial differential equations as well: The solution to the inhomogeneous DE is determined by the initial conditions, and a so-called *convolution integral* in which the integrand at time s, with $0 \le s \le t$, is a linear combination of homogeneous solutions at time $t - s$ $(t \ge t - s \ge 0)$ with the forcing function at time s. So, the history of the forcing before time t, combined with the homogeneous solutions at the specified related but different times, determines the inhomogeneous solution precisely in this particular way.

You can read more about the general *Duhamel's Principle* at Wikipedia and you might encounter it in more advanced applied mathematics or physics courses.

Friday application: We love the forced oscillator equation

$$x''(t) + x(t) = f(t).$$

For today's numerical experiment visualize a swing (pendulum) with the numerical value of $\frac{g}{L} = 1$. They used to have swings like that at Liberty Park - super long chains of around $10\ m$ in length. For such a linearized pendulum, i.e. small oscillations from the stable equilibrium, the natural angular frequency is $\omega_0 = 1$, and

$$x_H(t) = c_1 \cos(t) + c_2 \sin(t)$$

is simple harmonic motion of period $2\pi \approx 6.2$ seconds. Happy times.

Now, pretend you're a parent pushing your child on a swing, and you want to teach them about resonance. You did some work before going to the park and know that you don't have to only force with sinusoidal functions any more in order to figure out the response function $x(t)$. Besides, who pushes a swing like that anyways? In particular, you checked that if $x(t)$ solves the second order DE, then

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix} = \begin{bmatrix} x' \\ -x + f \end{bmatrix}.$$

So, $[x, x']^T$ solves the linear system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$

$$\underline{\mathbf{x}}(t) = e^{tA}\underline{\mathbf{x}}_0 + \int_0^t e^{(t-s)A}\underline{\mathbf{f}}(s)\, ds\ .$$

On Tuesday we computed

$$e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Exercise 1 Using the matrix exponential solution formula for the system, and the correspondence back to the second order DE, verify that the solution to the original second order IVP for the swing is

$$x(t) = x_0 \cos(t) + v_0 \sin(t) + \int_0^t f(s) \sin(t - s) \, ds$$

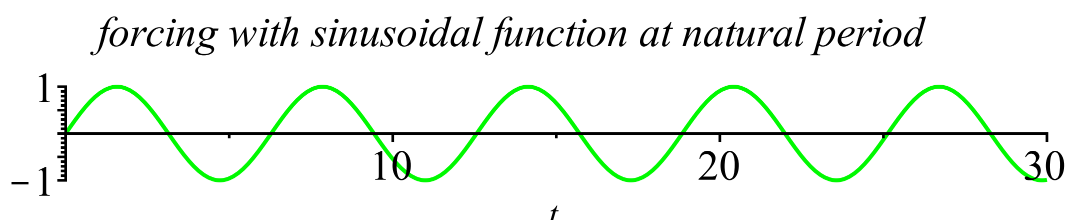
We are now ready to play the resonance game. We'll assume $x_0 = 0$, $v_0 = 0$ for simplicity.

Example 1

$$x''(t) + x(t) = \sin(t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

```
> with(plots):
  f1 := t→sin(t);
  plot1a := plot(f1(t), t = 0..30, color = green):
  display(plot1a, title = `forcing with sinusoidal function at natural period`);
```

$$f1 := t \mapsto \sin(t)$$



What's your vote? Will we get resonance?

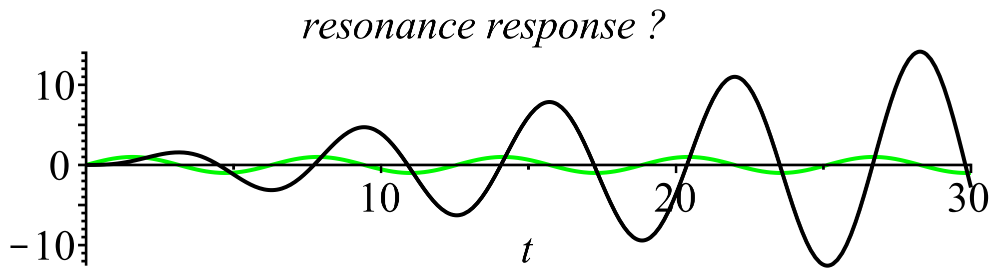
> $x1 := t \rightarrow \int_0^t \sin(\tau) \cdot f1(t - \tau)\, d\tau :$

$x1(t);$

$plot1b := plot(x1(t), t = 0 ..30, color = black) :$

$display(\{plot1a, plot1b\}, title = \text{`resonance response ?`});$

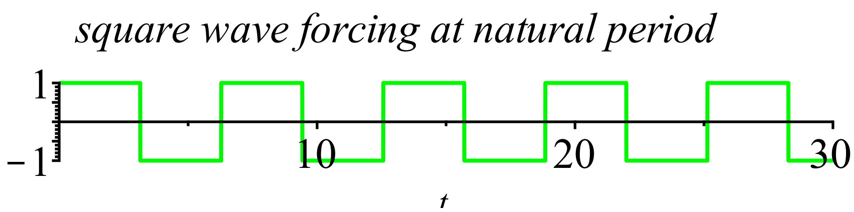
$$\frac{\sin(t)}{2} - \frac{\cos(t)\, t}{2}$$



Example 2) A square wave forcing function with amplitude 1 and period 2π, made with a linear combinations of unit step functions...

```
> with(plots):
> f2 := t -> -1 + 2·(sum((-1)^n·Heaviside(t - n·Pi), n = 0..10)):
  plot2a := plot(f2(t), t = 0..30, color = green):
  display(plot2a, title = `square wave forcing at natural period`);
```

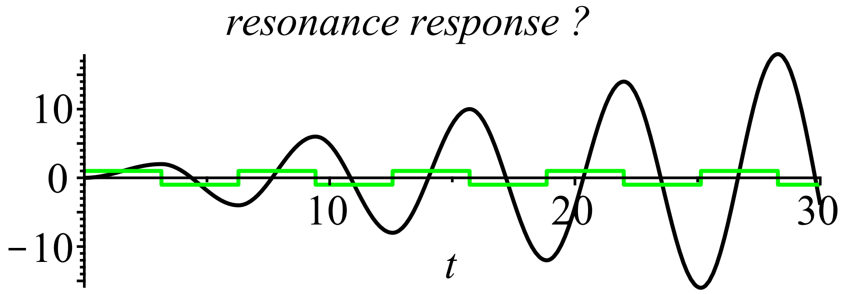
$$f2 := t \rightarrow -1 + 2\cdot\left(\sum_{n=0}^{10} (-1)^n \cdot \text{Heaviside}(t - n\cdot\text{Pi})\right):$$



1) What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

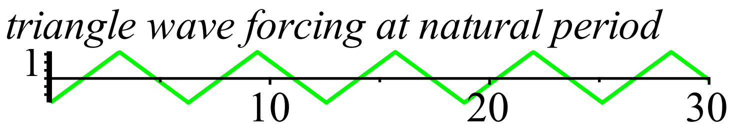
> $x2 := t \rightarrow \int_0^t \sin(\tau) \cdot f2(t - \tau) \, d\tau :$

$plot2b := plot(x2(t), t = 0 .. 30, color = black) :$
$display(\{plot2a, plot2b\}, title = `resonance\ response\ ?`);$



Example 3) A triangle wave forcing function, same period

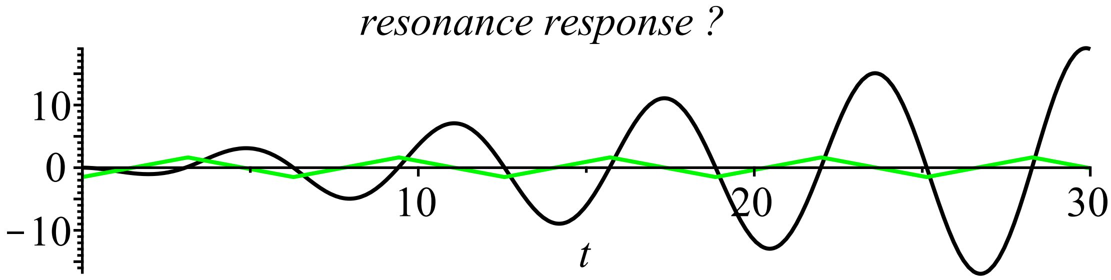
```
> f3 := t -> ∫_0^t f2(s) ds - 1.5 :  # this antiderivative of square wave should be triangle wave
plot3a := plot(f3(t), t = 0..30, color = green) :
display(plot3a, title = `triangle wave forcing at natural period`);
```



3) Resonance?

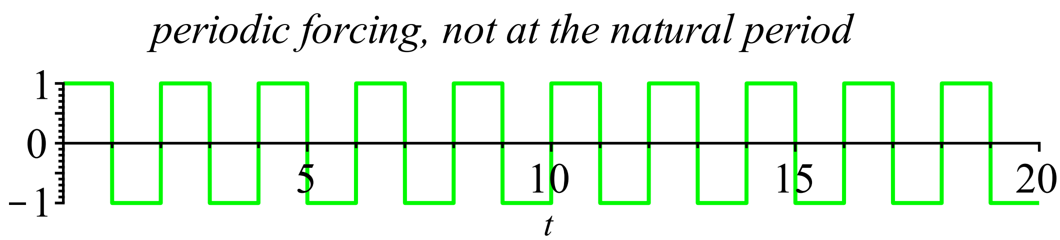
> $x3 := t \rightarrow \int_0^t \sin(\tau) \cdot f3(t - \tau)\, d\tau :$

$plot3b := plot(x3(t), t = 0 ..30, color = black) :$
$display(\{plot3a, plot3b\}, title = \text{`resonance response ?`});$



Example 4) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

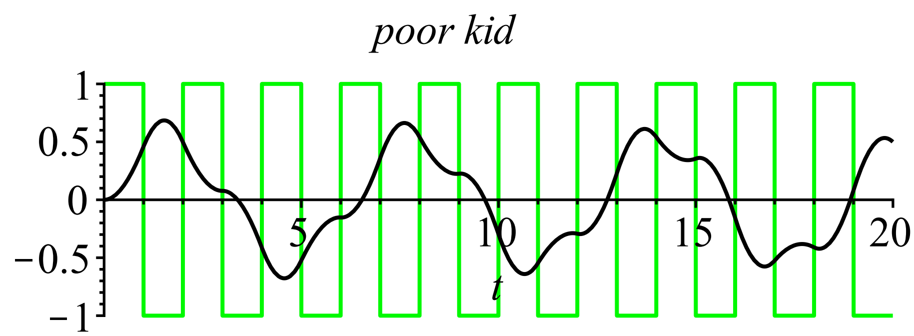
> $f4 := t \rightarrow -1 + 2 \cdot \sum_{n=0}^{20} (-1)^n \cdot \text{Heaviside}(t - n)$:
$plot4a := plot(f4(t), t = 0..20, color = green)$:
$display(plot4a, title = \text{`periodic forcing, not at the natural period`})$;



4) Resonance?

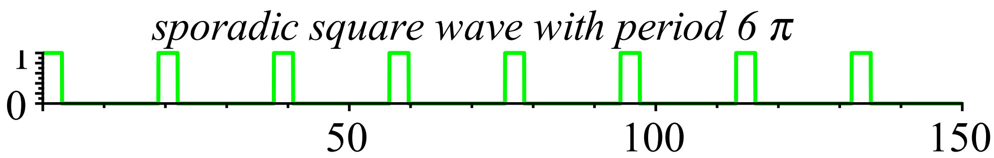
> $x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t-\tau) \, d\tau$:

plot4b := *plot*(*x4*(*t*), *t* = 0 ..20, *color* = *black*) :
display({*plot4a*, *plot4b*}, *title* = \`*poor kid* \`);



Example 5) Forcing *not* at the natural period, e.g. with a particular wave having period $T = 6\pi$.

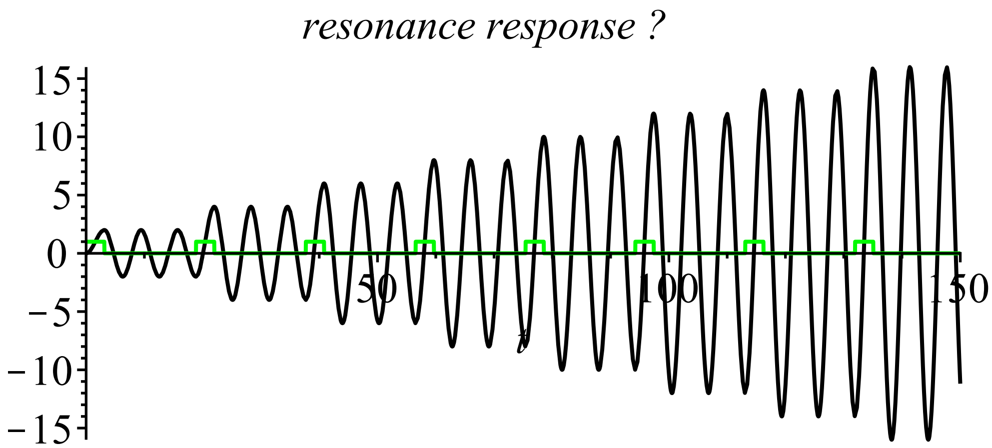
> $f5 := t \rightarrow \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi)\right)$:
$plot5a := plot(f5(t), t = 0..150, color = green)$:
$display(plot5a, title = \text{`sporadic square wave with period } 6\pi\text{`})$;



5) Resonance?

> $x5 := t \rightarrow \int_0^t \sin(\tau) \cdot f5(t - \tau) \, d\tau :$

$plot5b := plot(x5(t), t = 0..150, color = black) :$
$display(\{plot5a, plot5b\}, title = \text{`resonance response ?`});$



>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

__Precise Answer:__ It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\left\{1, \cos(\omega\, t), \sin(\omega\, t), \cos(2\,\omega\, t), \sin(2\,\omega\, t), \cos(3\,\omega\, t), \sin(3\,\omega\, t), \ldots\right\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which we will start studying on Monday next week. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0} = T_0$), there will be resonance; otherwise, no resonance.