Math 2280-002
Week 11, March 25-27 6.3-6.4, 5.4
Mon Mar 25
6.3-6.4 - further comments about conservative system critical point analysis; 6.5 - the increasing complexity of first order autonomous systems of differential equations as the number of DE's increases, and brief mention of chaotic systems.

Announcements:
- I've posted topic review notes for Friday exam (also at the end of Wed. notes)
- Thursday practice exam session 1:00-2:20 room TBA
- Today "overview" & further directions Chapter 6
  T,W § 5.4 (not on this test).

Warm-up Exercise: (you could look through the notes)

More precise summary of Friday discussions about conserved quantities for first order systems, and resulting saddle points and stable centers for the associated equilbrium points of the first order system. We consider a general first order autonomous system

$$\begin{cases} x'(t) = F(x, y) \\ y'(t) = G(x, y) \end{cases}$$

Theorem: Let $E(x, y)$ be any twice continuously differentiable function with non-degenerate critical points, that is associated to the first order system above in the sense that $E(x(t), y(t))$ is constant along each solution trajectory. (We call $E(x, y)$ a *conserved quantity* for the system of differential equations) Then critical points (x_* , y_*) for the first order systems, i.e. constant solutions, are precisely the points at which $\nabla E = [E_x, E_y] = \mathbf{0}$, i.e. multivariable calculus critical points for $E(x, y)$.

- If $E(x_* , y_*)$ is a local minimum value for E at which the graph $z = E(x, y)$ is concave up (i.e. the Hessian matrix of E is positive definite, i.e. has positive eigenvalues); or if $E(x_* , y_*)$ is a local maximum maximum value for E (i.e the Hessian matrix of E is negative definite, i.e. has negative eigenvalues) - then (x_* , y_*) is a stable center for the first order system of differential equations and the nearby level curves are nearly elliptical.

- If $(x_*, y_*, E(x_* , y_*))$ is a saddle point on the graph, then (x_*, y_*) is a saddle point for the first order nonlinear system.

(One can see why the theorem should be true, but a precise proof would require Math 3220-level analysis. Our two examples from Friday should make the theorem believable though.)

Example 2 from Friday: The second order differential equation for the freely-rotating rigid-rod pendulum,

$$\theta''(t) + \frac{g}{L}\sin\big(\theta(t)\big) = 0$$

arises from conservation of energy: The total energy

$$TE(t) = KE + PE = \frac{1}{2}mL^2\theta'(t)^2 + mgL\big(1 - \cos\big(\theta(t)\big)\big)$$

is constant once the pendulum is set in motion. Thus for the associated system that $[\theta(t), \theta'(t)]$ satisfies, namely

$$x'(t) = y \quad = F(x,y)$$
$$y'(t) = -\frac{g}{L}\sin(x) \quad = G(x,y)$$

the (rescaled) energy function

$$E(x,y) = \frac{1}{2}y^2 + \frac{g}{L}(1 - \cos(x)) \qquad \frac{TE}{mL^2}$$

is a conserved quantity. And

$$\nabla E = [E_x, E_y] = \left[\frac{g}{L}\sin(x), y\right] \quad = [0, 0].$$

has exactly the same zeroes ($y = 0$ and $x = n\pi$) as the first order system of differential equations. Furthermore:

constant sols.

$\sin\theta = 0$
$\theta = n\pi \quad n \in \mathbb{Z}.$

$\updownarrow$ equivalent since $x(t) = \theta$, $y'(t) = \theta'$

$y = 0$
$\sin x = 0$
$x = n\pi, \; n \in \mathbb{Z}$

$E(x,y)$ critical points

$\sin x = 0$
$y = 0$

same!!

Graph of the scaled energy function for rigid-rod pendulum, with $\frac{g}{L} = 1$:

$$E(x, y) = \frac{1}{2} y^2 + \frac{g}{L} (1 - \cos(x))$$

$\nabla E = [0,0] \iff y = 0$
$x = n\pi$

$E(x,y) \geq 0$ minima @ $(2\pi n, 0)$
x even mult of π

$(g/L = 1)$

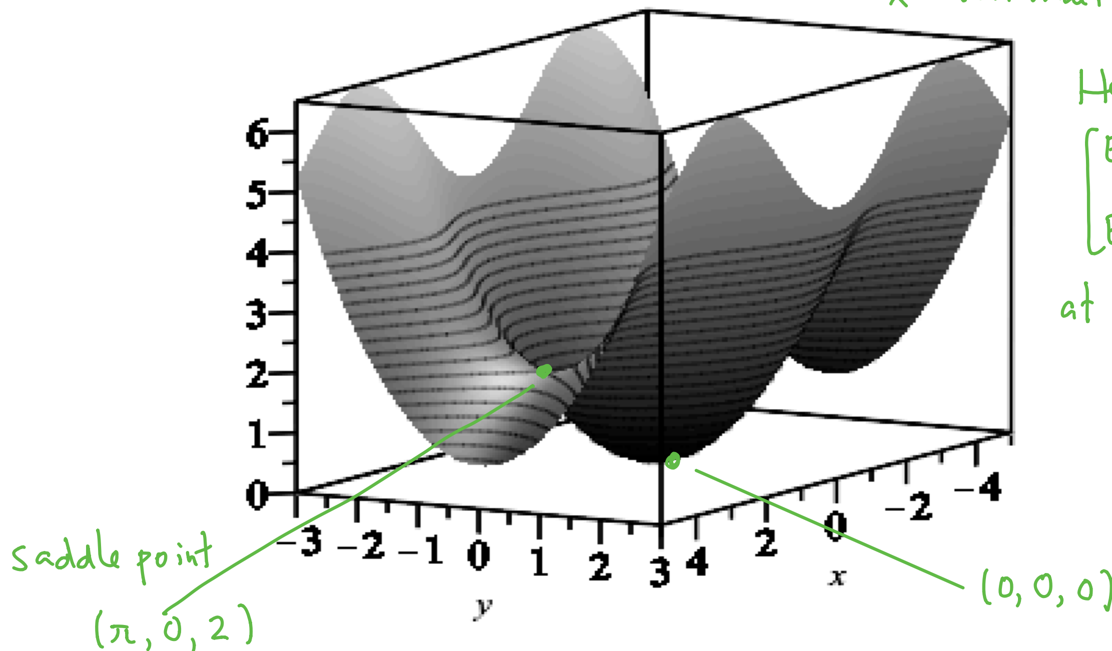
Hessian matrix for E

$$\begin{bmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{bmatrix} = \begin{bmatrix} \cos x & 0 \\ 0 & 1 \end{bmatrix}$$

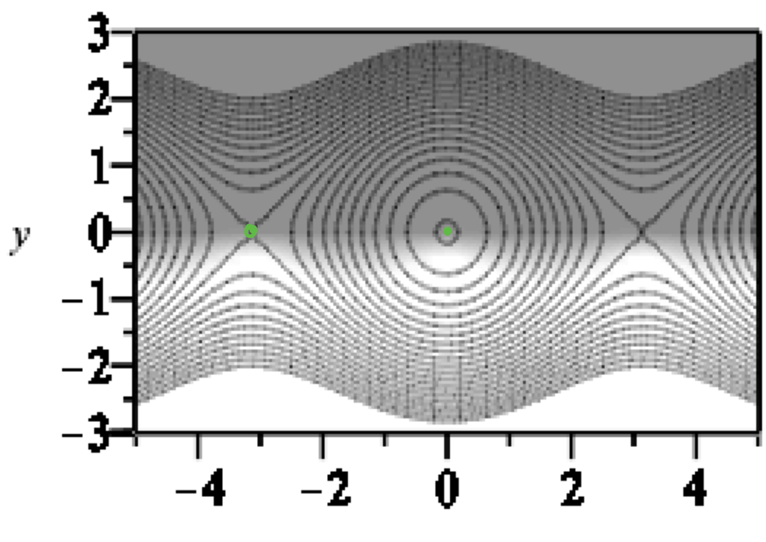
at even mult of π

Hessian is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
local min.

(0,0) Stable center



Top view, showing level curves (contours):

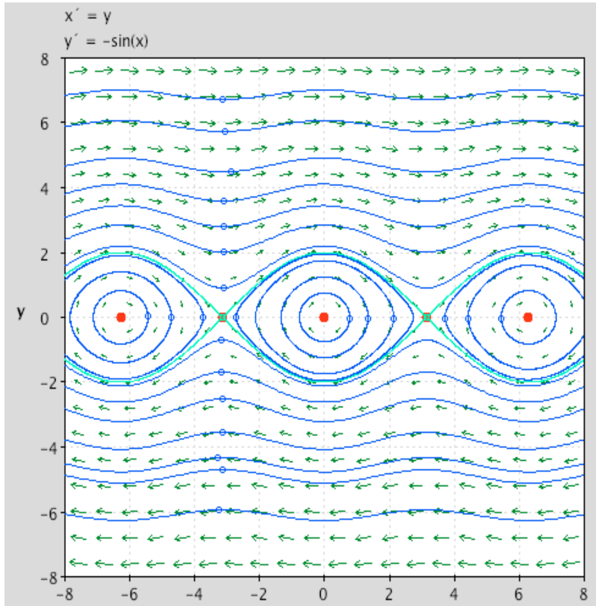


Hessian at every odd mult of π

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

saddle points.

pplane output:



Exercise 1 Even though we already know what the answers will be in the undamped case, let's work out the Jacobian matrices and linearizations at the equilibria for the undamped rigid rod pendulum system on the previous pages. Let's consider the possibility of damping at the same time.

even with damping same equil sol's

$$\theta''(t) + c\,\theta'(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

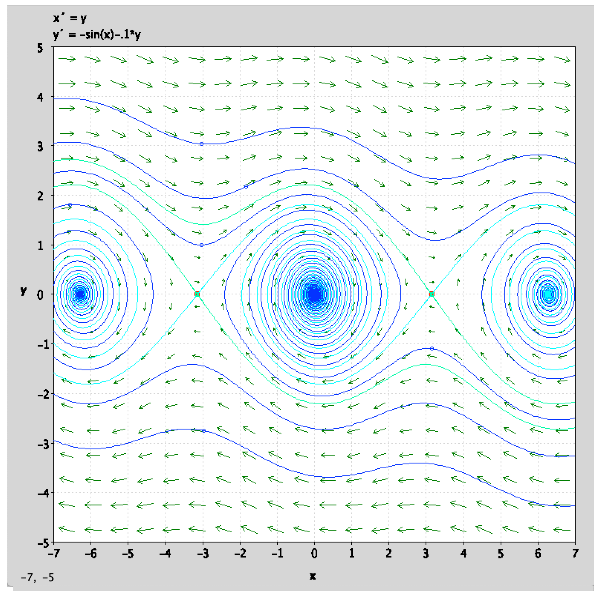
know how to do this $\begin{bmatrix}\theta\\ \theta'\end{bmatrix}$

θ const
$\sin\theta = 0$ again
$\theta = n\pi$

$$x'(t) = y \quad = F$$
$$y'(t) = -\frac{g}{L}\sin(x) - c\,y = G$$

$y = 0$
$-\frac{g}{L}\sin x - cy = 0$ $\Big\}$ $y = 0$, $\sin x = 0$, $x = n\pi$.

$$J_{@(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & -c \end{bmatrix}$$



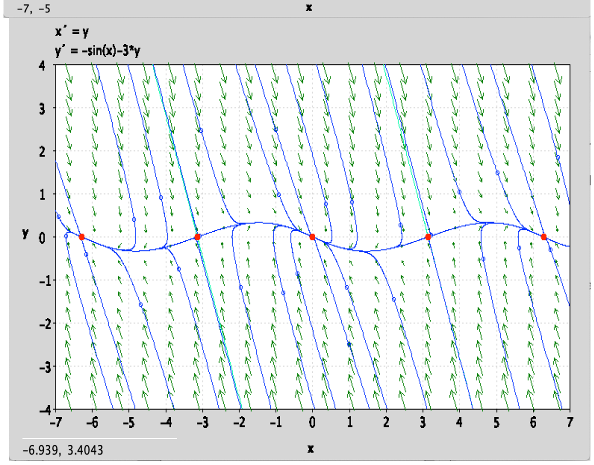
$\boxed{\begin{matrix} c = 0 \\ x = n\pi \\ y = 0 \end{matrix}}$: $J_{@(n\pi,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos n\pi & 0 \end{bmatrix}$

n even, $J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{L}$$

roots $\lambda = \pm i\sqrt{\frac{g}{L}}$

$x = n\pi$ n even
$y = 0$
linearization: stable center.



n odd. $J = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & 0 \end{bmatrix}$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ g/L & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{L}$$ roots $\pm\sqrt{\frac{g}{L}}$ saddle points

with damping

$$J_{@(n\pi,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos n\pi & -c \end{bmatrix}$$

n even $\begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -c-\lambda \end{vmatrix} = \lambda^2 + c\lambda + \frac{g}{L} = 0$

$$\lambda = \frac{-c \pm \sqrt{c^2 - \frac{4g}{L}}}{2}$$

n odd $J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -c \end{bmatrix}$

nodal sink overdamped. $\Big\{$ • $c^2 > \frac{4g}{L}$ two neg real roots

$|J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{L} & -c-\lambda \end{vmatrix} = \lambda^2 + c\lambda - \frac{g}{L} = 0$ underdamped $\left\{ c^2 < \frac{4g}{L} \right.$ complex roots negative real part

Saddles

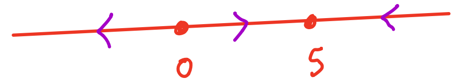
$\lambda = \frac{-c \pm \sqrt{c^2 + \frac{4g}{L}}}{2}$ spiral sinks

"Show and tell" of possible long-time behavior for solutions to systems of first order autonomous systems, depending on the number of differential equations in the system. We assume the systems satisfy the conditions for the existence-uniqueness theorem. It turns out we've only touched the surface of what can happen, because we've stayed in low dimensions.

$n = 1$. Let $x(t)$ solve a first order autonomous differential equation IVP

$$x'(t) = f(x)$$
$$x(0) = x_0.$$

$x'(t) = 3x(5-x)$



Then either

(1) $\lim_{t\to\infty} x(t) = x_e$, where x_e is an equilibrium solution, or

(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t\to t_1} |x(t)| = \infty$.

$n = 2$. Let $\underline{\mathbf{x}}(t) = [x(t), y(t)]^T$ solve the autonomous system of differential equations IVP

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$
$$x(0) = x_0$$
$$y(0) = y_0$$

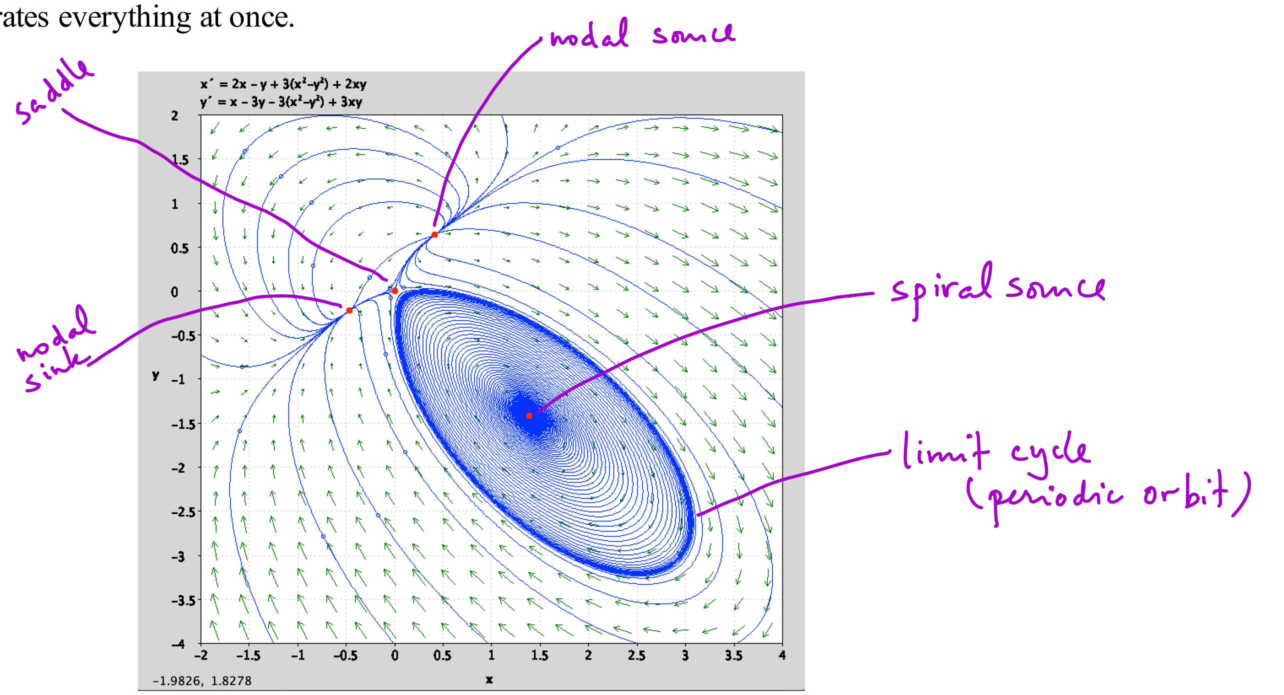
Then either

(1) $\lim_{t\to\infty} \underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_e$, where $\underline{\mathbf{x}}_e$ is an equilibrium solution, or

(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t\to t_1} |\underline{\mathbf{x}}(t)| = \infty$.

OR

(3) As $t\to\infty$ $\underline{\mathbf{x}}(t)$ converges to a *periodic limit cycle*. This is the mystery example *pplane* opens with, because it illustrates everything at once.



$n \geq 3$. Let $\underline{\boldsymbol{x}}(t) \subseteq \mathbb{R}^n$ solve a first order autonomous differential equation IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{f}}(\boldsymbol{x})$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 .$$

Then one of the following happens:

(1) $\lim_{t \to \infty} \underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_e$, where $\underline{\boldsymbol{x}}_e$ is an equilibrium solution, or

(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t \to t_1} \| \underline{\boldsymbol{x}}(t) \| = \infty$

(3) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ converges to a *periodic limit cycle.*

OR

(4) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ "converges" to a *strange attractor.*

(5) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ exhibits *chaotic behavior.*

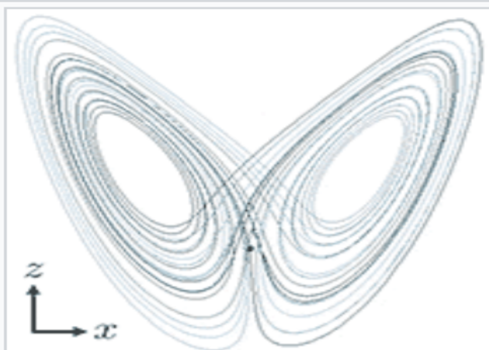
Our text explores some examples of (4),(5) in section 6.5. See also: Math 5410 *Introduction to ordinary differential equations* and Math 5470 *Chaos and non-linear systems.*

Example of strange attractor for the "Lorentz system"
https://en.wikipedia.org/wiki/Lorenz_system

Overview [edit]

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection.[1]

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z.$$



A sample solution in the Lorenz attractor when ρ = 28, σ = 10, and β = 8/3

Example of period doubling on the way to chaos, for the "forced Duffing equation":
https://en.wikipedia.org/wiki/Duffing_equation

This can be thought of as a (possibly damped) forced oscillation problem, for a mass on top of a wire (see text p. 432):

In your HW 6.4.14

$$x'' - 8x + 2x^3 = 0$$

$$x' = v$$

$$v' = 8x - 2x^3$$

$$= 2x(4 - x^2)$$

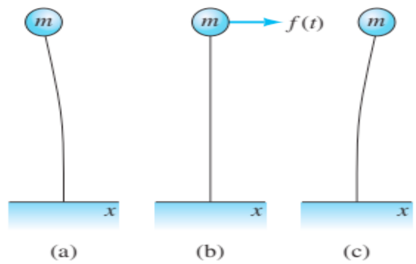


FIGURE 6.5.12. Equilibrium positions of a mass on a filament:
(a) stable equilibrium with $x < 0$;
(b) unstable equilibrium at $x = 0$;
(c) stable equilibrium with $x > 0$.

$$x''(t) + \delta\, x'(t) + \alpha\, x(t) + \beta\, x^3(t) = \gamma \cos(\omega\, t)$$

This can be thought of as an *autonomous* system of three first order DE's, for $[x(t), x'(t),\ t]$, in the time variable e.g. τ:

$$x_1'(\tau) = x_2$$

$$x_2'(\tau) = -\alpha\, x_1 - \beta\, x_1^3 - \delta\, x_2 + \gamma \cos(\omega\, \tau)$$

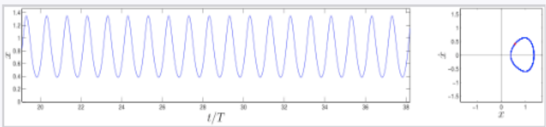
$$t'(\tau) = 1$$

$$x''(t) + \delta x'(t) + \alpha x(t) + \beta x^3(t) = \gamma \cos(\omega t)$$

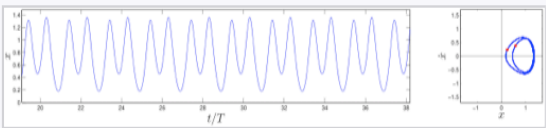
Examples [edit]

Some typical examples of the time series and phase portraits of the Duffing equation, showing the appearance of subharmonics through period-doubling bifurcation – as well chaotic behavior – are shown in the figures below. The forcing amplitude increases from $\gamma = 0.20$ to $\gamma = 0.65$. The other parameters have the values: $\alpha = -1$, $\beta = +1$, $\delta = 0.3$ and $\omega = 1.2$. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$. The red dots in the phase portraits are at times t which are an integer multiple of the period $T = 2\pi/\omega$.[10]

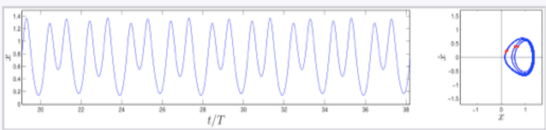
Time traces and phase portraits



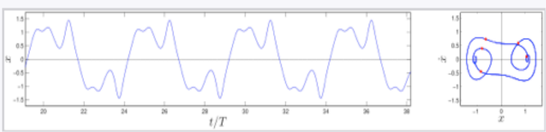
period-1 oscillation at $\gamma = 0.20$



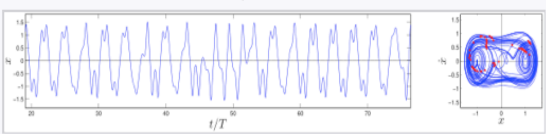
period-2 oscillation at $\gamma = 0.28$



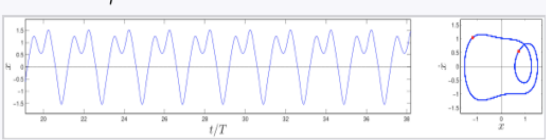
period-4 oscillation at $\gamma = 0.29$



period-5 oscillation at $\gamma = 0.37$



chaos at $\gamma = 0.50$



period-2 oscillation at $\gamma = 0.65$

Tues Mar 26
5.4 mass-spring systems

Announcements:

- practice exam session Thursday 1-2:20 LCB 323
- w10.3 typos ... corrections circled.

(1)
$$x' = a_1 x - b_1 x^2 - c_1 x y$$
$$y' = a_2 y - b_2 y^2 - c_2 x y$$

(2) first quadrant equil:
$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix}\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

- text typo eqtn (7) on page 406 is mis-copied from previous page. Should read
$$x' = 2x - xy$$
$$y' = -5y + xy$$

Warm-up Exercise: Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Could you use it to sketch the phase portrait for $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$?
(although that's not related to today's lecture).

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = (\lambda+2+1)(\lambda+2-1) = (\lambda+3)(\lambda+1)$$

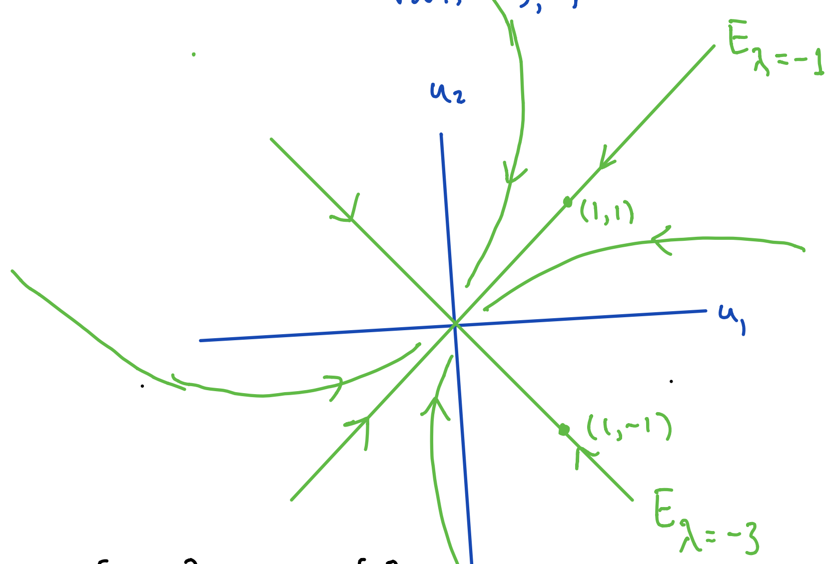
roots $-3, -1$

$E_{\lambda=-1}$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$E_{\lambda=-3}$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$= \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.



$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

nodal sink

on exam, be able to do this for all flavors of non-zero eigendata

5.4 Mass-spring systems: untethered mass-spring trains, and forced oscillation non-homogeneous problems.

Consider the mass-spring system below, with no damping. Although we draw the picture horizontally, it would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.

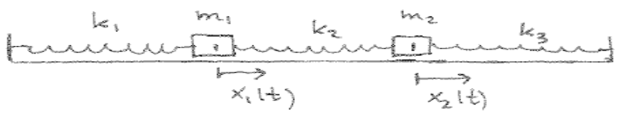
$m_1 x_1'' = \Delta F_{\text{spring } 1} + \Delta F_{\text{spring } 2}$

$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$

$m_2 x_2'' = \Delta F_{\text{spring } 2} + \Delta F_{\text{spring } 3}$

$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$

think about the signs!



Let's make sure we understand why the natural system of DEs and IVP for this system is

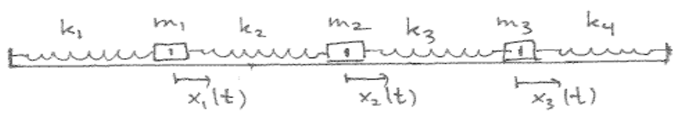
$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why? (Hint: after deriving the system of second order differential equations write down an equivalent system of first order differential equations.)

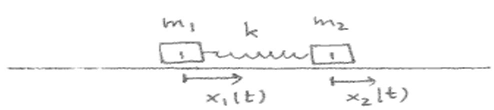
dim = 4 because 4 IC's uniquely determine sol'n.

OR this system is equivalent to a 1st order linear homog system for x_1, $v_1 = x_1'$, x_2, $v_2 = x_2'$

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:



dim sol'n space 6 ($2n$)



4

$$\cancel{m_1} x_1'' = -k_1 x_1 + k_2(x_2 - x_1) = -\frac{(k_1+k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2$$
$$\cancel{m_2} x_2'' = -k_2(x_2 - x_1) - k_3 x_2 = \frac{k_2}{m_2} x_1 - \frac{(k_2+k_3)}{m_2} x_2$$

We can write the system of DEs for the system at the top of the previous page in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 3 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\boxed{\underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}}.}$$

(You can think of A as the "acceleration" matrix.)

Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}}, \text{ with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{r\,t}\underline{\boldsymbol{v}}$$

We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function. But let's try the substitution directly, in analogy to what we did for higher order single linear differential equations back in Chapter 3.

Now, in the present case of systems of masses and springs we are assuming there is no damping. Thus, the total energy - consisting of the sum of kinetic and potential energy - will always be conserved. Any two complex solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{r\,t}\boldsymbol{v}^{\pm} = e^{(a\,\pm\,\omega i)t}\boldsymbol{v}^{\pm}$$

would yield two real solutions $\underline{\boldsymbol{X}}(t), \underline{\boldsymbol{Y}}(t)$ where

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{X}}(t) \pm i\,\underline{\boldsymbol{Y}}(t)\ .$$

Because of conservation of energy ($TE = KE + PE$ must be constant), neither $\underline{\boldsymbol{X}}(t)$ nor $\underline{\boldsymbol{Y}}(t)$ can grow or decay exponentially - if a solution grew exponentially the total energy would also grow exponentially; if it decayed exponentially the total energy would decay exponentially. SO, we must have $a = 0$. In other words, in order for the total energy to remain constant we must actually have

$$\underline{\boldsymbol{x}}(t) = e^{i\,\omega\,t}\underline{\boldsymbol{v}}\ .$$

Substituting this $x(t)$ into the homogeneous DE

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

yields the necessary condition

$$-\omega^2\cancel{e^{i\,\omega\,t}}\underline{\boldsymbol{v}} = \cancel{e^{i\,\omega\,t}}A\,\underline{\boldsymbol{v}}\quad .$$

So $\underline{\boldsymbol{v}}$ must be an eigenvector, with non-positive eigenvalue $\lambda = -\omega^2$,

$$A\,\underline{\boldsymbol{v}} = -\omega^2\,\underline{\boldsymbol{v}}\ .$$

And since row reduction will find real eigenvectors for real eigenvalues, we can find eigenvectors $\underline{\boldsymbol{v}}$ with real entries. And the two complex solutions

$$\underline{\boldsymbol{x}}(t) = e^{\pm\,i\omega t}\underline{\boldsymbol{v}} = \cos(\omega\,t)\underline{\boldsymbol{v}} \pm i\sin(\omega\,t)\underline{\boldsymbol{v}}$$

yield the two real solutions

$$\underline{\boldsymbol{X}}(t) = \cos(\omega\,t)\underline{\boldsymbol{v}},\qquad \underline{\boldsymbol{Y}}(t) = \sin(\omega\,t)\underline{\boldsymbol{v}}\ .$$

$\vec{v}$ is eigenvector

But : $\lambda = -\omega^2$

$(\omega = \sqrt{-\lambda})$

So, we skip the exponential solutions altogether, and go directly to finding homogeneous solutions of the form above. We just have to be careful to remember that $\underline{\boldsymbol{v}}$ is an eigenvector with eigenvalue $\lambda = -\omega^2$, i.e. $\omega = \sqrt{-\lambda}$.

Note: In analaogy with the scalar undamped oscillator DE

$$x''(t) + \omega_0^2\, x = 0$$

where we could read off and check the solutions

$$\cos\left(\omega_0 t\right),\ \sin\left(\omega_0 t\right)$$

directly without going throught the characteristic polynomial, it is easy to check that

$$\cos(\omega\, t)\underline{\boldsymbol{v}},\ \sin(\omega\, t)\underline{\boldsymbol{v}}$$

each solve the conserved energy mass spring system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

as long as

$$-\omega^2\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}.$$

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

If $A_{n\times n}$ is a diagonalizable matrix and if all of its eigenvalues are negative, then for each eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ there are two linearly independent solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j$$

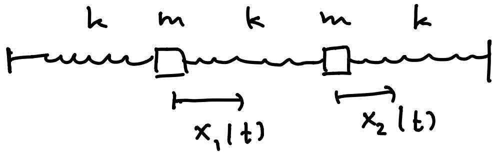
with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

warmup!



$$m x_1'' = -k x_1 + k(x_2 - x_1)$$
$$m x_2'' = -k(x_2 - x_1) - k x_2$$

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

warm-up

$$E_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$
$$E_{\lambda=-3} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

b) Eigenvectors same, but evals multiplied by $\frac{k}{m}$

$$\underbrace{\frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\frac{k}{m}\left(-1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$$

$$E_{\lambda=-\frac{k}{m}} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \qquad \omega_1 = \sqrt{\frac{k}{m}}$$
$$E_{\lambda=-3\frac{k}{m}} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

soltn
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\underbrace{(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_1(t) \equiv x_2(t)}}_{\text{"in-phase" mode}} + \underbrace{(c_3 \cos\omega_2 t + c_4 \sin\omega_2 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x_1(t) \equiv -x_2(t)}$$

out of phase mode

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos\left(\omega_1 t - \alpha_1\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos\left(\omega_2 t - \alpha_2\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos\left(\omega_1 t\right) + c_2 \sin\left(\omega_1 t\right)\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos\left(\omega_2 t\right) + c_4 \sin\left(\omega_2 t\right)\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

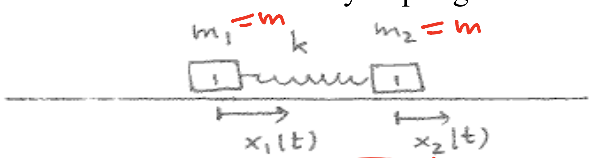
Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

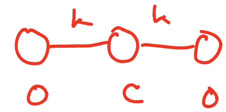
$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 4) Consider a train with two cars connected by a spring:

relates to CO_2 problem





stretch amt $x_2 - x_1$

4a) Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

4b). Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems for next week.

$$\cancel{m}\, x_1''(t) = \frac{k}{m}(x_2 - x_1)$$
$$\cancel{m}\, x_2''(t) = -\frac{k}{m}(x_2 - x_1)$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 1 = (\lambda+1+1)(\lambda+1-1) = (\lambda+2)\lambda$$

roots $\lambda = 0, -2$

so for $\frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

eigenvals $\lambda = 0, -\frac{2k}{m}$

continue next week...

Wed Mar 27
5.4 mass-spring systems, and forced oscillations

Announcements:

- practice exam 1–2:20 tomorrow LCB 323
- end of class today – go over topics (you can questions)
- start with warmup exercise.

  review of 2nd order sys → experiment

  finish Tuesday notes.

Warm-up Exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~ e.g. homog tank problem

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~ two mass, three spring problem.

Eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

$$E_{\lambda=-5} = span\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}, \quad E_{\lambda=-1} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

1a) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}' = A\vec{x}$ (1)

build soln space out of vector fcns $e^{\lambda t}\vec{v}$ where λ = eval $A\vec{v} = \lambda\vec{v}$.

try $\vec{x}(t) = e^{\lambda t}\vec{v}$ in (1)

$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$

$A\vec{x} = Ae^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$

$= e^{\lambda t}\lambda\vec{v}$

1b) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$+ (c_3\cos t + c_4\sin t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(2) $\vec{x}'' = A\vec{x}$

$A\vec{v} = \lambda\vec{v}$ $(\lambda < 0)$

$\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$ solve (2), for $\underline{\omega = \sqrt{-\lambda}}$ $\omega^2 = -\lambda$

e.g. if $\vec{x}(t) = \cos\omega t\,\vec{v}$

then $\vec{x}''(t) = -\omega^2\cos\omega t\,\vec{v}$ & $A\cos\omega t\,\vec{v} = \cos\omega t\,A\vec{v}$

$= \cos\omega t\,\lambda\vec{v}$

$= -\omega^2\cos\omega t\,\vec{v}$

Math 2280
Experiments
March 27, 2019

We're using the same apparatus as we used for our single mass-spring experiment. In that experiment we measured that an additional mass of $50\ g$ caused the spring to stretch $15.8\ cm$, and we checked that if we added another $50\ g$ mass, the stretch amount was almost the same - verifying that these springs from Physics have roughly the same Hooke's constant at varying lengths.

So, solving the equation $k\,x = m\,g$ for k we have

> $k := \frac{.05 \cdot 9.806}{.158};$

$$k := 3.103164557 \tag{1}$$

The slow (in-phase) mode has

> $m := .05;$
> $w_1 := \sqrt{\frac{k}{m}};$
> $T_1 := \frac{2 \cdot \pi}{w_1};$ *#period in seconds*

$$m := 0.05$$
$$w_1 := 7.877816956$$
$$T_1 := 0.7975794998 \tag{2}$$

20 cycles
17.1
16.9
16.9
17.2
17.1
17.05

$\frac{17.05}{20} \approx .853$

in phase

The fast (out of phase) mode has

> $w_2 = \sqrt{3.} \cdot w_1;$
> $T_2 = \frac{T_1}{\sqrt{3.}};$ *#period in seconds*

$$12.95105130 = 13.64477922$$
$$T_2 = 0.4604827388 \tag{3}$$

50 cycles
22.9
23.1
23
23.0

$\frac{23}{50} \approx .46$ seconds

!!

out of phase

Probably our experiment will run slow....

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 3.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator back in Chapter 3 notes, assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A-matrix with the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(The "M" matrix turns out to not be diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m := .05;
  ms := .011;
  k := 3.103;
```

$$w_1 := \sqrt{\frac{6\cdot k}{6\cdot m + 5\cdot ms}};$$

$$w_2 := \sqrt{\frac{6\cdot k}{2\cdot m + ms}};$$

$$T_1 := evalf\left(\frac{2\cdot \mathrm{Pi}}{w_1}\right);$$

$$T2 := evalf\left(\frac{2\cdot \mathrm{Pi}}{w_2}\right);$$

$$m := 0.05$$
$$ms := 0.011$$
$$k := 3.103$$
$$w_1 := 7.241896880$$
$$w_2 := 12.95105130$$
$$T_1 := 0.8676159592 \qquad \text{exp } .853 \text{ sec.}$$
$$T2 := 0.4851486696 \qquad .46 \qquad (4)$$

better... I was expecting even better, though ☺

>

Forced oscillations (still undamped):

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\ .$$

If the forcing is sinusoidal,

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\underline{\boldsymbol{G}}_0$$
$$\Rightarrow\ \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\underline{\boldsymbol{F}}_0$$

with $\underline{\boldsymbol{F}}_0 = M^{-1}\underline{\boldsymbol{G}}_0$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)\ ,$$

and we've been discussing how to find the homogeneous solutions $\underline{\boldsymbol{x}}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega\, t)\ \underline{\boldsymbol{c}}$$

where the vector $\underline{\boldsymbol{c}}$ is what we need to find.

Exercise 2) Substitute the guess $\underline{\boldsymbol{x}}_P(t) = \cos(\omega\, t)\ \underline{\boldsymbol{c}}$ into the DE system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\underline{\boldsymbol{F}}_0$$

to find a matrix algebra formula for $\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:

$$\underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}}_0\ .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 3) Continuing with the configuration from ~~Monday's~~ Tuesday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1, \omega_2 = \sqrt{3}$ and that

$$\underline{\boldsymbol{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos\left(\sqrt{3}\, t - \alpha_2\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\boldsymbol{x}}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

Exam 2 Review
Math 2280-002
Spring 2019

Exam 2 is Friday March 29 from 12:50-1:50 p.m. in our classroom LCB 219. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.

The test will cover sections 3.5-3.6, 4.1, 5.1-5.3, 6.1-6.4, and some of the questions may also tie in to material we covered earlier in the course. Higher likelyhood topics are underlined, and all listed topics are possible unless ruled out explicitly. Percentage weights below add up to more than 100% because exam questions may touch on more than one chapter.

I will post at least one practice test, and will work through it in a problem session Thursday March 28, 2:00-3:20 in a room to be announced.

Chapter 3) 3.5-3.6 (at least 25% of exam)

- 3.5: Finding particular solutions y_p to solve $L(y) = f$, then using the complete solution $y = y_p + y_H$ to solve initial value problems.
    - Undetermined coefficients either in math examples, or in mass-spring oscillation examples from 3.6.

- 3.6 Forced oscillation problems:
    - undamped phenomena: superposition with homogeneous solution, beating, resonance
    - damped phenomena: steady periodic and transient solutions; practical resonance. amplitude-phase form.
    - using conservation of energy TE=PE+KE to derive differential equations of motion for mass-spring and pendulum configuration.

Chapter 4) 4.1 (at least 20% of exam)

- 4.1 Systems of differential equations
    - existence-uniqueness theorem for systems of first order DE's
    - how to convert a second order (or higher order) DE IVP to an equivalent first order system of DE's IVP, and the equivalences between the two frameworks.
    - modeling input-output systems (e.g. tanks) for solute amounts in each compartment

Chapter 5) 5.1-5.5 (at least 40% of the exam)

- 5.1 Theory for linear systems of differential equations:
  - why the solution to $L(y) = f$ is $y = y_P + y_H$
  - why solution space to $L(y) = 0$ is a subspace, and what its dimension is (based on existence-uniqueness theorem).
  - Calculus differentiation rules for sums and products of vector and matrix valued functions.
- 5.2 Eigenvalue-eigenvector method for solving $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$ (Most naturally coupled with an input-output problems or first order system versions of Chapter 3 mass-spring problems).
  - diagonalizable case with real eigenvalues or complex eigenvalues
  - solving $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$ for simple $\underline{\boldsymbol{f}}$, either with $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ or via change of functions using Math 2270 diagonalization identities.
- 5.3 reproducing and classifying phase portraits for $x' = A\,x$ when $n = 2$ using eigendata and general solutions to sketch phase portraits with real eigenvalues; using eigenvalues and sampling tangent field along coordinate axes to sketch portraits in case of complex eigenvalues.

Chapter 6) 6.1-6.4 (at least 25% of the exam)

- 6.1-6.2 Identifying equilibria of first order systems of two autonomous differential equations algebraically. Using linearization and eigendata from Jacobian matrices to classify the type of equilibrium solution, understand the implications for stability, and to be able to sketch what the phase portrait looks like near the equilibrium solution. Interpreting pplane phase portraits.
- 6.3 population models, and what the various terms in the model represent.
- 6.4 nonlinear mechanical models, e.g pendulum and nonlinear springs. Using conservation of energy (or other conserved quantities e.g in section 6.3) in cases where linearization near an equilibrium point is indeterminant, in order to deduce a stable center for the nonlinear problem.