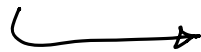Math 2280-002
Week 10, March 18-22 5.3, 6.1-6.4
Mon Mar 18
5.3 - phase portraits for homogeneous systems of two linear DE's - summary of complex eigendata case from Friday, and discussion of real eigendata examples; 6.1 Introduction to systems of two autonomous first order differential equations.

<u>Announcements:</u> Exam 2 next Friday ☺

take an time, understand shapes of phase portraits
as relates to eigendata of $A_{2\times2}$

<u>Warm-up Exercise:</u> no warmup today, but if you missed Friday
maybe scan page 3 of today's
notes → we understood n=2
homogeneous phase portraits
with $\mathbb{C}$ eigendata.

5.3 phase portraits for two linear systems of first order differential equations

$$\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A. This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points. On Friday before break we analyzed the case of complex eigendata. Today we'll analyze what happens with real eigendata.

Here's a summary of what happens if the matrix A (with real entries) has complex eigendata:

complex eigenvalues (from Friday before break, included here for completeness): Let $A_{2 \times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{v} = \underline{a} + \underline{b}\,i$. Then we know that we can use the complex solution $e^{\lambda t}\underline{v}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\underline{z}(t) = e^{\lambda t}\underline{v} = e^{(p + q\,i)t}(\underline{a} + \underline{b}\,i)$$
$$= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{a} + \underline{b}\,i)$$
$$= \left[e^{p\,t}\cos(q\,t)\underline{a} - e^{p\,t}\sin(q\,t)\underline{b}\right]$$
$$+ i\left[e^{p\,t}\sin(q\,t)\underline{a} + e^{p\,t}\cos(q\,t)\underline{b}\right] .$$

Thus, the general <u>real</u> solution is a linear combination of the real and imaginary parts of the solution above. I put the linear combination weights c_1, c_2 on the right instead of the left in the expression below, to facilitate seeing the matrix factorization in the next step:

$$\underline{x}(t) = e^{p\,t}\left[\cos(q\,t)\underline{a} - \sin(q\,t)\underline{b}\right]c_1$$
$$+ e^{p\,t}\left[\sin(q\,t)\underline{a} + \cos(q\,t)\underline{b}\right]c_2 .$$ weights

We can rewrite $\underline{x}(t)$ as

$$\underline{x}(t) = e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

$$\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p\,t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.
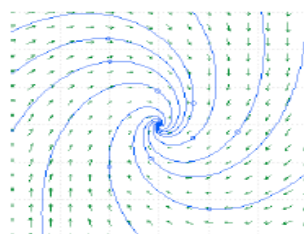
Thus $\underline{x}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:
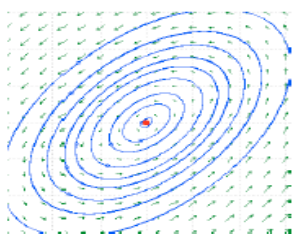
center
Re(λ)=0
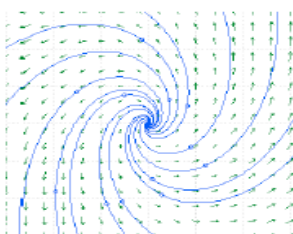
spiral source
Re(λ)>0

spiral sink
Re(λ)<0

<u>Real eigenvalues</u> If the matrix $A_{2 \times 2}$ is <u>diagonalizable,</u> i.e. if there exists a basis $\{\underline{v}_1, \underline{v}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of A), then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\underline{x}' = A\underline{x}$ is
$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2$$

- And, for each $\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2$ the value of the tangent field at x is
$$A\underline{x} = A(c_1 \underline{v}_1 + c_2 \underline{v}_2) = c_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \underline{v}_2.$$

(The text discusses the case of non-diaganalizable A. This can only happen if $det(A - \lambda I) = (\lambda - \lambda_1)^2$, but the $\lambda = \lambda_1$ eigenspace is defective so that its dimension is one instead of two.)

<u>Exercise 1)</u> Here is an example of what happens when A has <u>two real eigenvalues of opposite sign</u>.
Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = -2$
$\lambda_2 = 3$

$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

a) The eigendata for A is

$$E_{\lambda=-2} = span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \qquad E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

$A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = c_1 A(\vec{v}_1)$
$+ c_2 A\vec{v}_2$
$= c_1 \lambda_1 \vec{v}_1 +$
$c_2 \lambda_2 \vec{v}_2$

b) Use just the eigendata to sketch the tangent vector field on the plot below. Begin by sketching the two eigenspaces.   See hex + page

c) Use the general solutions to the DE system to overlay representative solution curves. Notice that your work in b, c is consistent.

$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
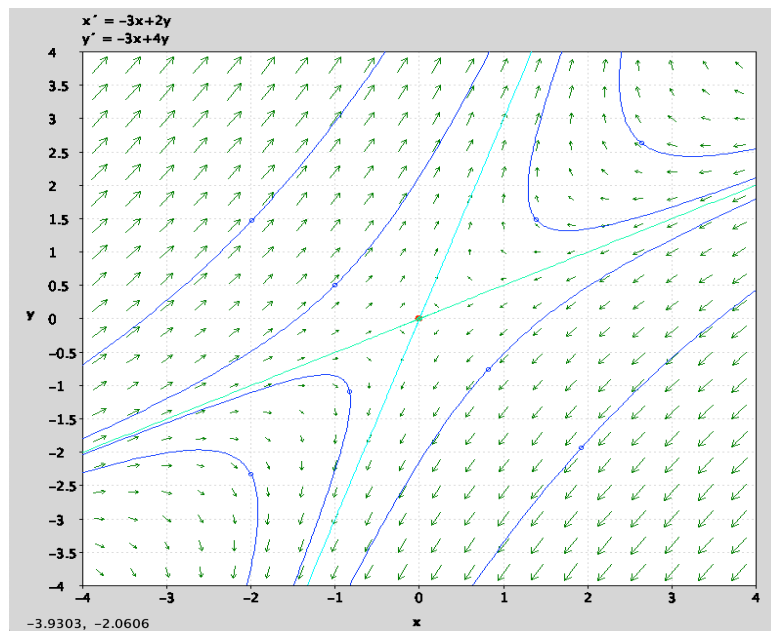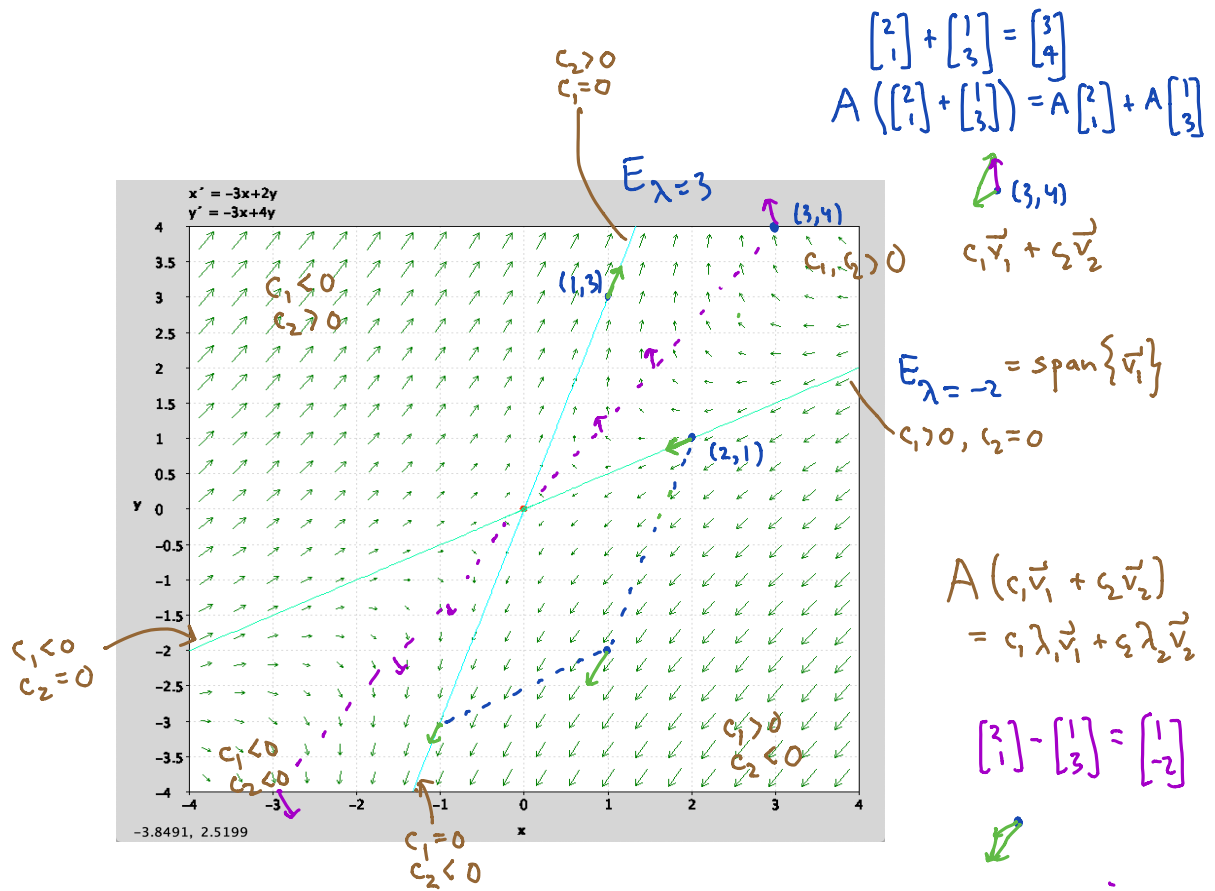
if $c_1 \neq 0, c_2 = 0$
we're in $E_{\lambda=-2}$
$\lim_{t \to \infty} \vec{x}(t) = \vec{0}$

if $c_1 = 0, c_2 \neq 0$
$\lim_{t \to \infty} \vec{x}(t) = \infty$
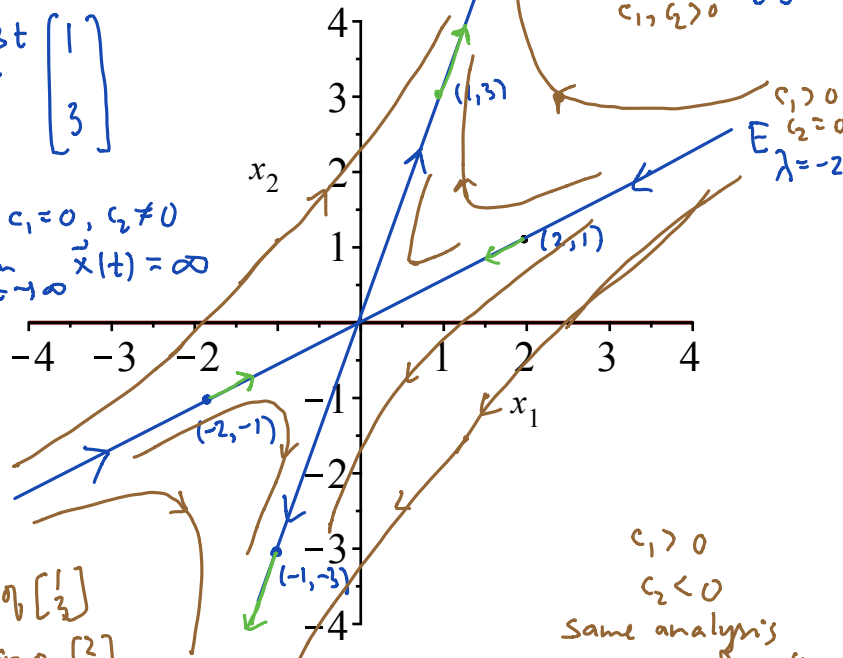
e.g.
$c_1, c_2 > 0$
$\lim_{t \to \infty} \vec{x}(t) = \vec{0} + \vec{\infty}$
$\uparrow$
$\lim_{t \to -\infty} \vec{x}(t) = \vec{\infty}$ in dir of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
in dir of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

any mult of sol is sol

$c_1, c_2 > 0$

$c_1 > 0$
$c_2 < 0$
Same analysis
then scale & reflect

x' = -3x+2y
y' = -3x+4y

$c_2 > 0$
$c_1 = 0$

$E_{\lambda=3}$

$(3,4)$

$c_1, c_2 > 0$

$c_1 < 0$
$c_2 > 0$

$(1,3)$

$E_{\lambda=-2} = \text{span}\{\vec{v_1}\}$

$c_1 > 0, \ c_2 = 0$

$(2,1)$

$c_1 < 0$
$c_2 = 0$

$c_1 < 0$
$c_2 < 0$

$c_1 = 0$
$c_2 < 0$

$c_1 > 0$
$c_2 < 0$

-3.8491, 2.5199

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$A\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = A\begin{bmatrix} 2 \\ 1 \end{bmatrix} + A\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$(3,4)$

$c_1\vec{v_1} + c_2\vec{v_2}$

$$A\left(c_1\vec{v_1} + c_2\vec{v_2}\right) = c_1\lambda_1\vec{v_1} + c_2\lambda_2\vec{v_2}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

x' = -3x+2y
y' = -3x+4y

-3.9303, -2.0606

Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign.
Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Use the eigendata of A to find the general solution to the first order system of DE's.

$$E_{\lambda=3} = span\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}, \quad E_{\lambda=6} = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$
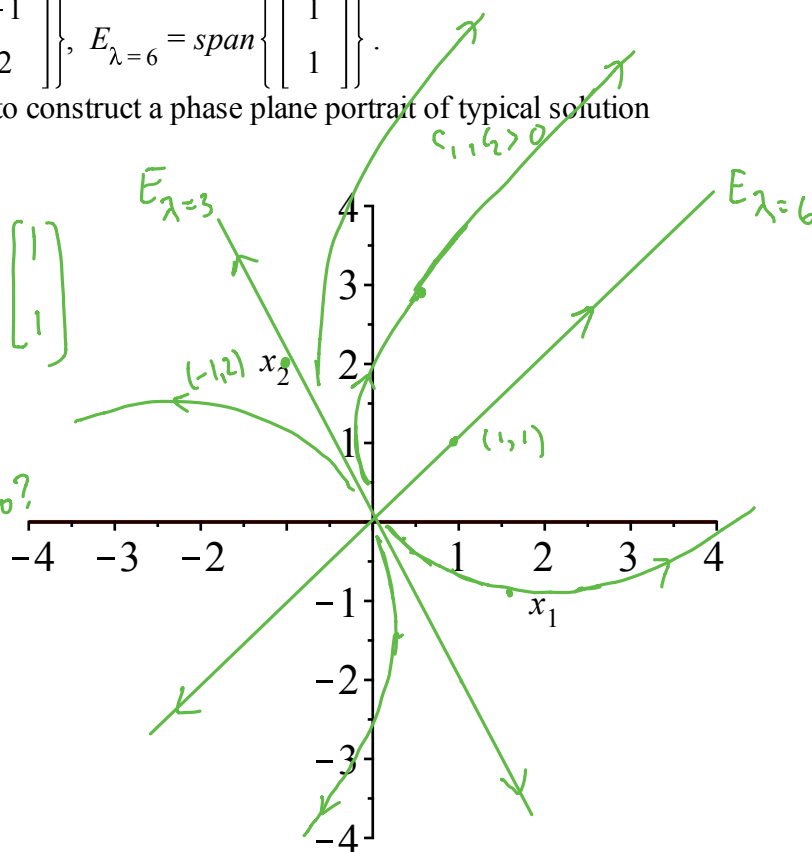
b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves. First sketch the eigenspaces.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$c_1, c_2 > 0$

which term dominates

$\vec{x}(t), \vec{x}'(t)$ as $t \to \infty$?

$c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

as $t \to -\infty$ which term

decays slowest?

$c_1 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$E_{\lambda=3}$

$c_1, c_2 > 0$

$E_{\lambda=6}$

$(-1,2)$ x_2

$(1,1)$

x_1

$x' = 5x+y$
$y' = 2x+4y$

$-2.8118, -4.6649$
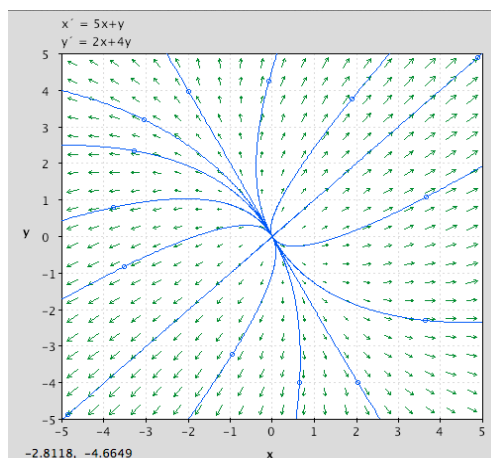
Theorem: Time reversal: If $\underline{x}(t)$ solves
$$\underline{x}' = A\,\underline{x}$$
then $\underline{z}(t) := \underline{x}(-t)$ solves
$$\underline{z}' = (-A)\underline{z}$$
proof: by the chain rule,
$$\underline{z}'(t) = \underline{x}'(-t)\cdot(-1) = -\underline{x}'(-t) = -A\,\underline{x}(-t) = -A\,\underline{z}.$$

Exercise 3)
a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A?
b) Describe how the eigendata of the matrix in the system below is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits and solutions are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
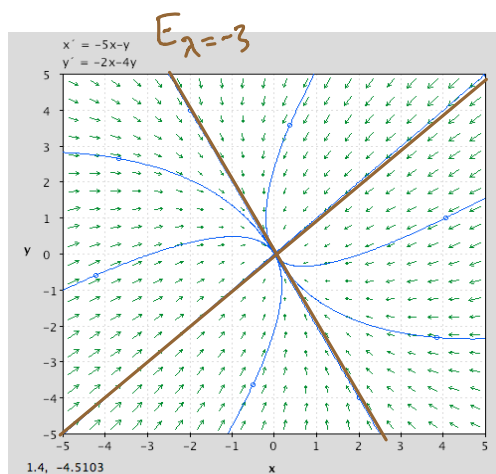
a) If $A\vec{v} = \lambda\vec{v}$

then $(cA)\vec{v} = c\lambda\vec{v}$

• mult matrix by c
   same eigenvectors, but
      eigenvalue $c\lambda$

b)   So if $c = -1$, eigenvalues switch
                          sign,

   so the arrows on phase portrait reverse,
   and the solution curves go in
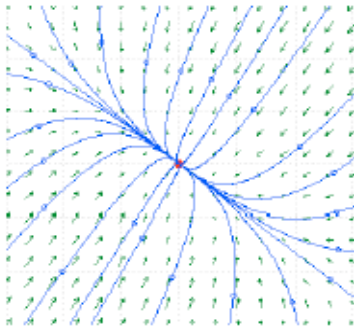   the opposite directions

x' = -5x-y
y' = -2x-4y

$E_{\lambda=-3}$

$E_{\lambda=-6}$

1.4, -4.5103

<u>summary</u>: In case the matrix $A_{2 \times 2}$ is <u>diagonalizable with real number eigenvalues</u>, the first order system of DE's
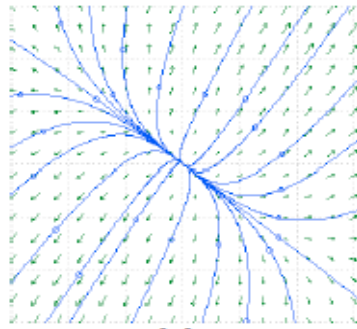
$$\underline{x}'(t) = A\,\underline{x}$$

has general solution

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2 \, .$$

If each eigenvalue is non-zero, the three possibilities are:

nodal sink	nodal source	saddle point
$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
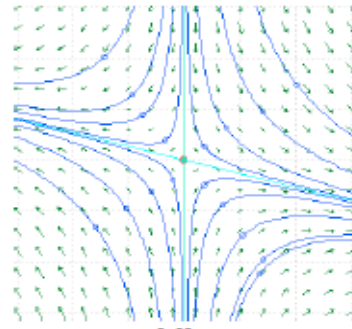
Could you indentify the eigenspaces, and which ones are $E_{\lambda=\lambda_1}$, $E_{\lambda=\lambda_2}$?

<u>Introduction to Chapter 6.</u> (We'll return to Chapter 5 after we discuss Chapter 6.) This chapter is about general (non-linear) systems of two first order differential equations for $x(t), y(t)$, i.e.
$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$
which we often abbreviate, by writing
$$x' = F(x, y, t)$$
$$y' = G(x, y, t) .$$
If we assume further that the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t , i. e.
$$x' = F(x, y)$$
$$y' = G(x, y)$$
then we call such a system <u>autonomous.</u> Autonomous systems of first order DEs are the focus of most of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2.

<u>Constant solutions</u> to an autonomous differential equation or system of DEs are called <u>equilibrium solutions.</u> Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system

$$F(x, y) = 0$$
$$G(x, y) = 0$$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems
$$x' = F(x, y)$$
$$y' = G(x, y)$$
are called <u>stable</u> if solutions to IVPs <u>starting close (enough)</u> to $[x_*, y_*]^T$ <u>stay as close as desired.</u>
- Equilibrium solutions are <u>unstable</u> if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called <u>asymptotically stable</u> if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.
(Notice these definitions are completely analogous to our discussion in Chapter 2.)

<u>stable</u> : $\forall \, \varepsilon > 0$ $\exists \, \delta > 0$ s.t. $\left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} x_* \\ y_* \end{bmatrix} \right\| < \delta$

(that's how close)
you to stay)

(how close
you need to
start)

then $\left\| \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} x_* \\ y_* \end{bmatrix} \right\| < \varepsilon$

for all $t > 0$.

<u>Exercise 4)</u> Consider the "competing species" model from section 6.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

logistic   competition

$$x'(t) = 14\,x - 2\,x^2 - xy \quad = F$$
$$y'(t) = 16\,y - 2\,y^2 - xy . \quad = G \checkmark$$

<u>a)</u> Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other). Hint: to understand why this model is plausible for $x(t)$ consider the normalized birth rate ~~rate~~ $\dfrac{x'(t)}{x(t)}$, as we did in Chapter 2.

b) $14x - 2x^2 - xy = 0 = x\,(14 - 2x - y)$

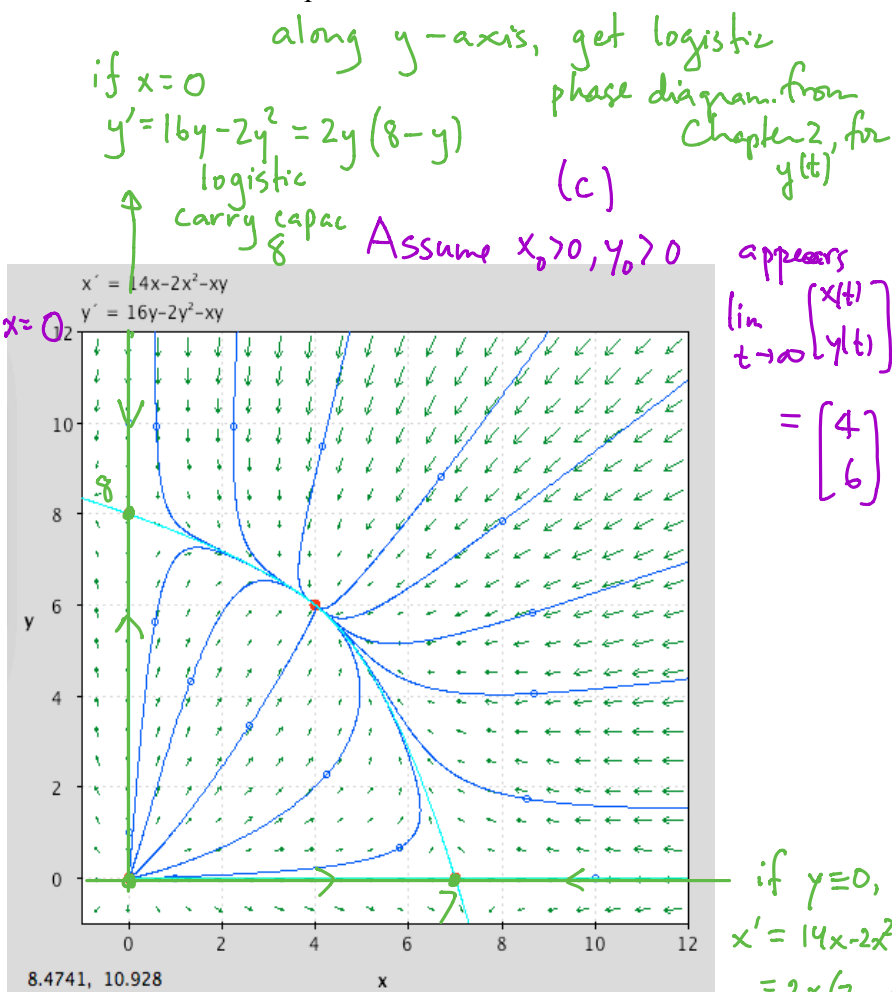 $16y - 2y^2 - xy = 0 = y\,(16 - 2y - x)$

<u>b)</u> Find the four equilibrium solutions to this competition model, algebraically.
<u>c)</u> What does the phase portrait below indicate about the dynamics of this system?
<u>d)</u> Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability, based on what the phase portrait looks like near each equilibrium solution?

$X = 0$ or $14 - 2x - y = 0$

and   and   and   and

$y = 0$ $16 - 2y - x = 0$ $y = 8$ $16 - 2y - x = 0$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$2x + y = 14$
$x + 2y = 16$
Solve!

if $x = 0$
$y' = 16y - 2y^2 = 2y(8-y)$
   logistic
   carry capac
     8

along y-axis, get logistic phase diagram from Chapter 2, for $y(t)$

(c)

Assume $x_0 > 0, y_0 > 0$ appears

$\displaystyle \lim_{t \to \infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

x' = 14x - 2x² - xy
y' = 16y - 2y² - xy

8.4741, 10.928

if $y = 0$,
$x' = 14x - 2x^2$
   $= 2x(7-x)$
   get logistic phase diagram for $x(t)$, Chptr 2

Tues Mar 19
6.1-6.2 Autonomous systems of two first order differential equations; linearization near equilibrium
solutions.

• for linearization, need to recall key idea from
differential multivariable Calc

• Monday Chapter 6 | here!
  - Tuesday   ↓

Warm-up Exercise: Suppose the temperature downtown (say at Temple Square)
is currently 45°. And suppose that nearby, it
decreases about .2°/mile as you go north, and .5°/mile
as you go east.
Approximate the temperature at the airport, which is
about 8 miles west and 2 miles north of T.S.

$$T(A.P.) \approx 45° + (.5°/\text{mile west})(8 \text{ miles west}) - (.2°/\text{mile north})(2 \text{ miles north})$$

$$= 45 \quad + \quad 4 \quad - .4$$
$$= 48.6°$$

$$T(A.P.) \approx T(T.S.) + \frac{\partial T}{\partial E}(T.S.)(\Delta E) + \frac{\partial T}{\partial N}(\Delta N)$$

-.5°/mile (-8 miles)   -.2°/mile . 2 miles .

Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math
2280 course. (The "linear drag" velocity model, Newton's law of cooling, small oscillation pendulum
motion, and the damped spring equation were all linearizations.) It's important to understand how to
linearize in general, because linearized differential equations can often be used to understand stability and
solution behavior near equilibrium points, for the original differential equations. Today we'll talk about
linearizing *systems* of DE's, which we've not done before in this course.

An easy case of linearization in <u>Exercise 4</u> from Monday's notes is near the equilibrium solution $[x_*, y_*]^T = [0, 0]^T$. It's pretty clear that our rabbit-squirrel population system

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

linearizes to

$$x'(t) = 14x \qquad x(t) = c_1 e^{14t}$$
$$y'(t) = 16y \qquad y(t) = c_2 e^{16t}$$

i.e.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:

nodal source.

x' = 14x−2x²−xy
y' = 16y−2y²−xy

Sadl

8.4741, 10.928
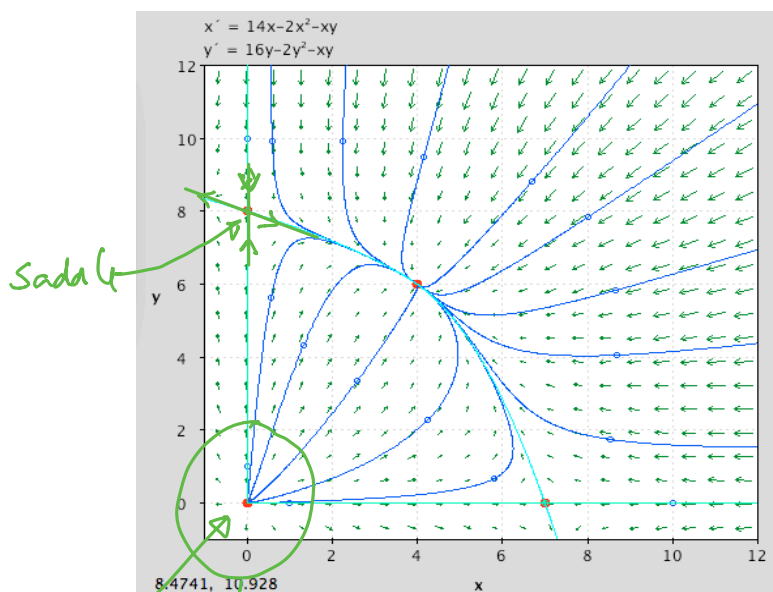
nodal source

like magnified

x' = 14x
y' = 16y

−0.0074074, 0.10206
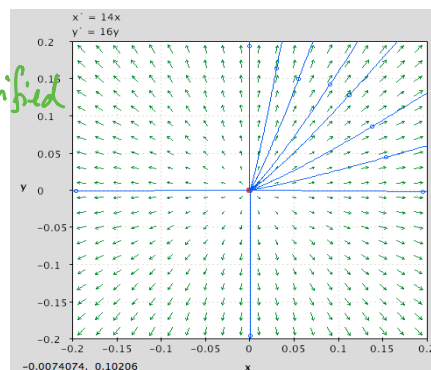
We use multivariable Calculus to linearize at equilibrium points that are not the origin. (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter.)
Here's how: Consider the autonomous system

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \ .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \ .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*) u + \frac{\partial F}{\partial y}(x_*, y_*) v + \varepsilon_1(u, v)$$

$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*) u + \frac{\partial G}{\partial y}(x_*, y_*) v + \varepsilon_2(u, v)$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$.
Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ .$$

The matrix of partial derivatives is called the <u>Jacobian matrix</u> for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.
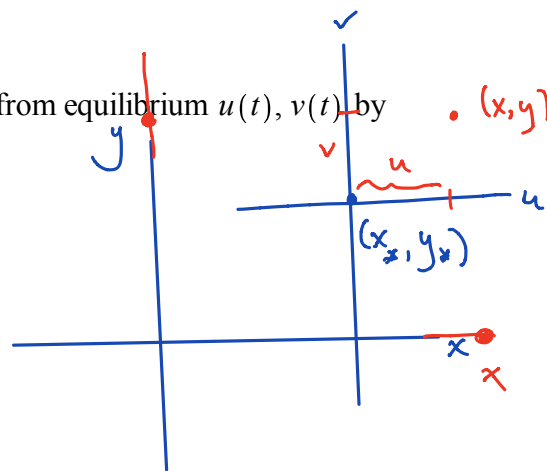
Space to review the multivariable tangent approximation, which is *the* key concept behind multivariable differential calculus, as well as being important for change of variables in multiple variable integration.

<u>Exercise 1)</u> We will linearize the rabbit-squirrel (competition) model of the running example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy \quad = F(x, y)$$
$$y'(t) = 16y - 2y^2 - xy \quad = G(x, y)$$

<u>1a)</u> Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from $4, 6$, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

$$\begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \checkmark$$

$$J_{@(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 14 - 4x - y & -x \\ -y & 16 - 4y - x \end{bmatrix}. \qquad J_{@(4,6)} = \begin{bmatrix} 14 - 16 - 6 & -4 \\ -6 & 16 - 24 - 4 \end{bmatrix}$$

<u>1b)</u> The matrix in the linear system of DE's above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{v}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{v}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, the phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$.

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \cong c_1 e^{-4.7t} \begin{bmatrix} .79 \\ -.64 \end{bmatrix} + c_2 e^{-15.3t} \begin{bmatrix} .49 \\ -.89 \end{bmatrix}$$

nodal sink,
asymptotically stable

$E_{\lambda = \lambda_2}$

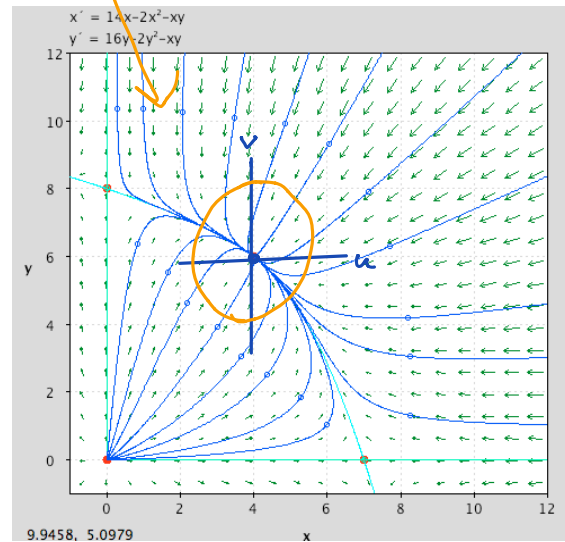$(.49, .89)$

$(.79, -.64)$

$E_{\lambda = \lambda_1}$
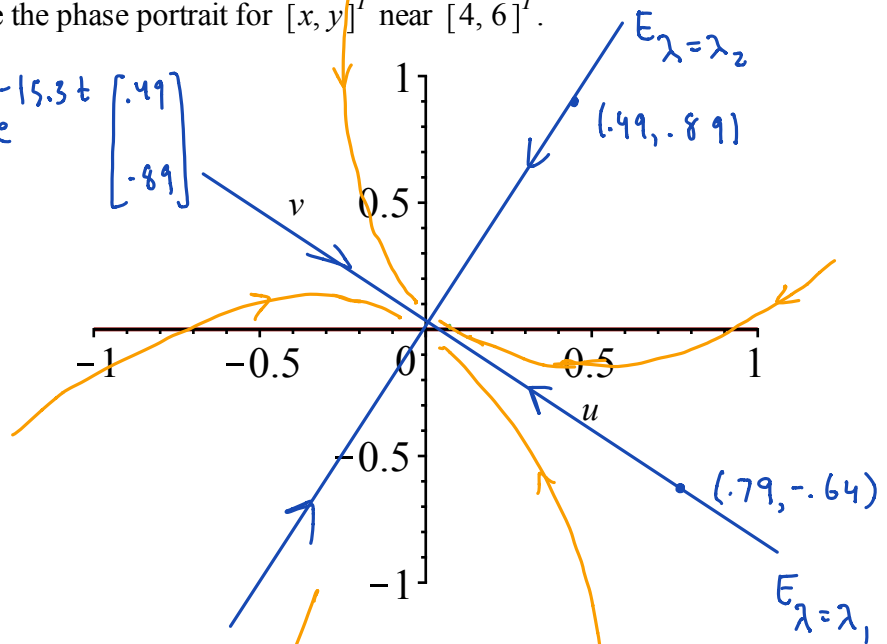
note all other orange
curves are dilations
of these

compare

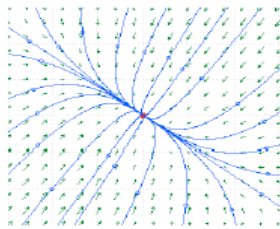Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

The non-linear problem and representative solution curves:

$x' = 14x - 2x^2 - xy$
$y' = 16y - 2y^2 - xy$

9.9458, 5.0979

pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilbrium solution" option under the "solution" menu item.

Equilbrium Point:
  There is a nodal sink at (4, 6)
  Jacobian:
        −8   −4
        −6   −12
  The eigenvalues and eigenvectors are:
        −4.7085   (0.77218, −0.63541)
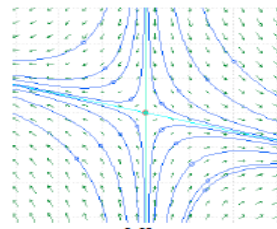   −15.292   (0.48097, 0.87674)

The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.
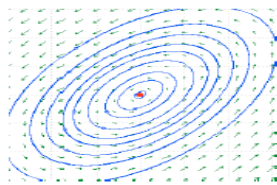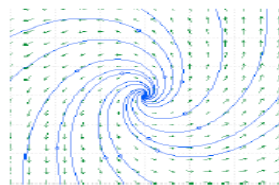

nodal sink	nodal source	saddle point
$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
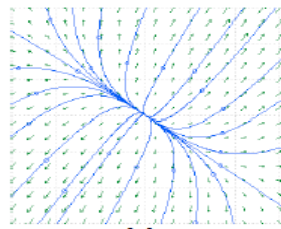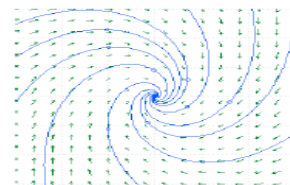
| center	spiral source	spiral sink
$\mathrm{Re}(\lambda)=0$	$\mathrm{Re}(\lambda)>0$	$\mathrm{Re}(\lambda)<0$

<u>Theorem:</u> Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.
(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.
(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Fun examples of borderline cases where the linearization at the origin has purely imaginary eigenvalues, so the origin is a stable center for the linearization but all three flavors for the three nonlinear systems:

stable center
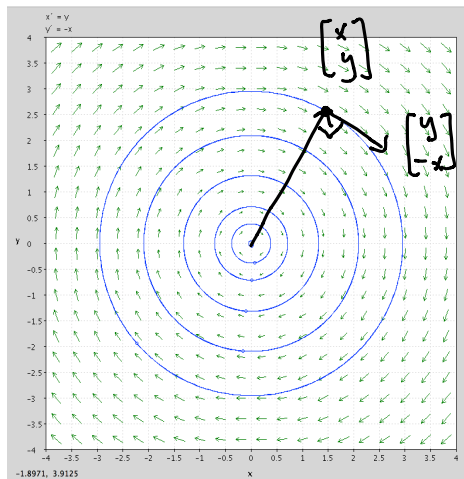
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
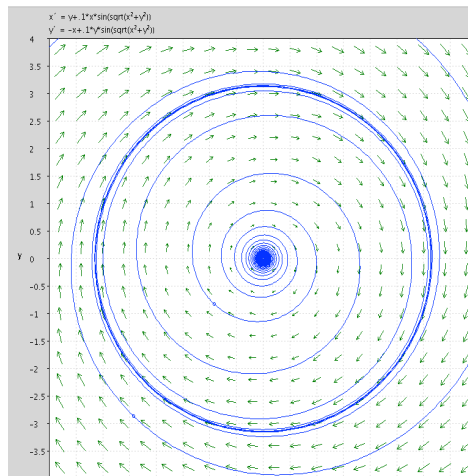
note $x'' = y' = x$

$x'' + x = 0$

$p(r) = r^2 + 1$

$x(t) = A\cos(t-\alpha)$

$x'(t) = y = -A\sin(t-\alpha)$

orbits circles.

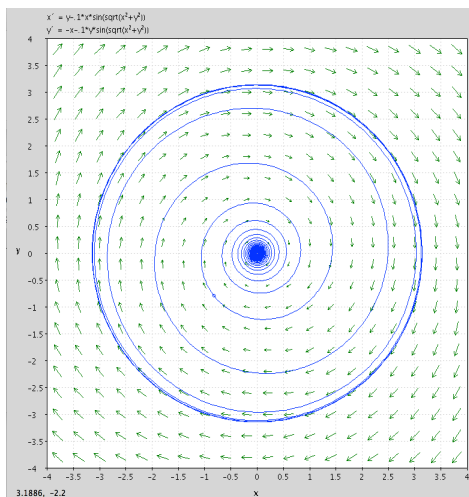$|A - \lambda I| = \lambda^2 + 1$

roots

$\lambda = \pm i$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} + \underbrace{.1\sin\sqrt{x^2+y^2}}_{\text{positive}} \begin{bmatrix} x \\ y \end{bmatrix}$$

as before $0 < \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| < \pi$

results in spiral
source
because of higher order

outward
radial

Wed Mar 20
6.3 Ecological models, continued

Warm-up Exercise: This is a "predator - prey" model. Which is which?
Also, what are the equilibrium solutions?

$$x'(t) = ax - pxy$$
$$y'(t) = -by + qxy$$

$a, b, p, q > 0.$

prey : $-pxy$ says $x(t)$ doesn't benefit from $y(t)$
      $+qxy$ says y' is greater if there is more x

ax if no y, then $x' = ax$ → grows exponentially
$-by$ if there's no x, y dies off exponentially.

So $x(t)$ is the prey
      $y(t)$ is the predator.

There are many interesting two-species models. In class and in one of your homework problems we've considered examples of the *logistic competition model* between two species:

logistic | competition

$$x'(t) = a_1 x - b_1 x^2 - c_1 x y$$
$$y'(t) = a_1 y - b_2 y^2 - c_2 x y$$

Here the constants $a_1, a_2, b_1, b_2, c_1, c_2$ are all positive. It turns out that if the logistic inhibition, as measured by the product $b_1 b_2$ is stronger than the competitive pressure as measured by $c_1 c_2$, i.e.

$$b_1 b_2 > c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always asymptotically stable. This is what happened in our class example

$$x'(t) = 14 x - 2 x^2 - x y$$
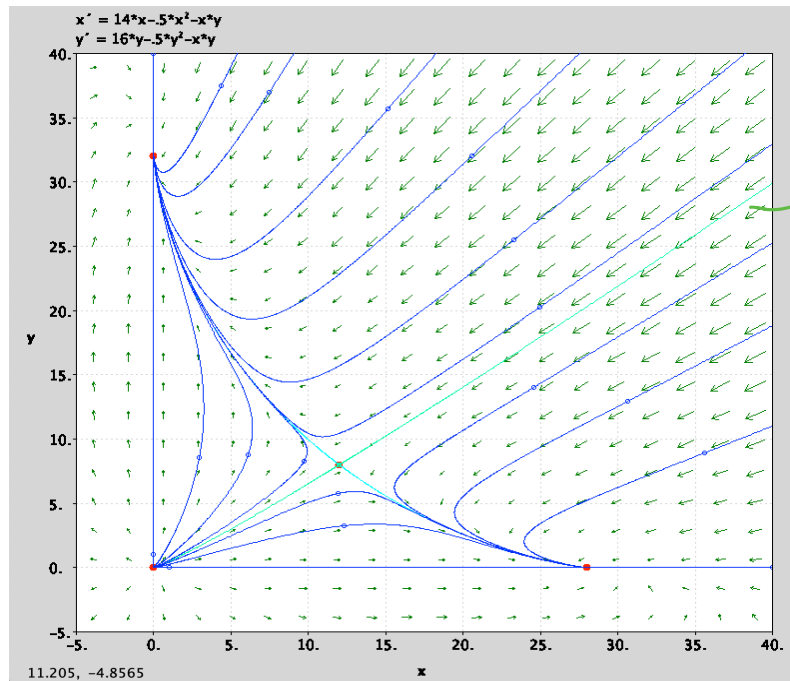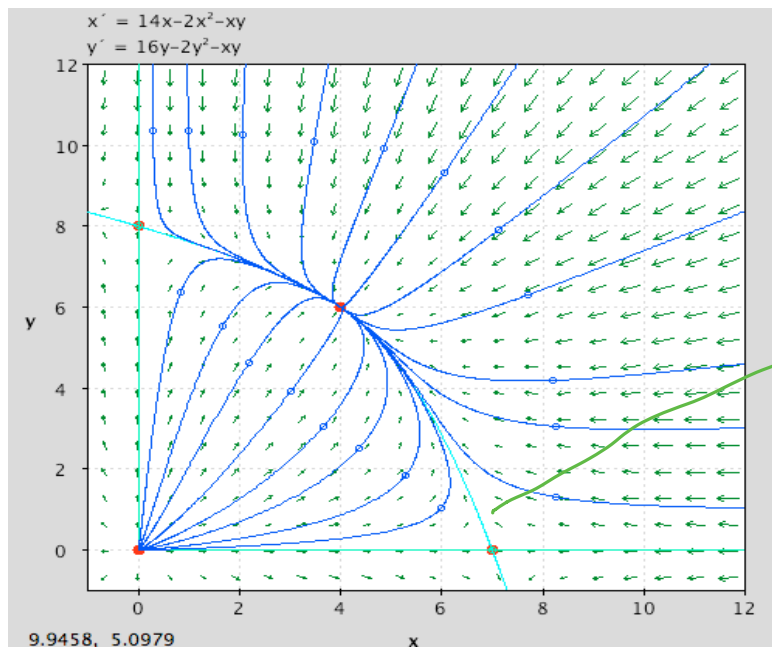$$y'(t) = 16 y - 2 y^2 - x y.$$

On the other hand, if

$$b_1 b_2 < c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always unstable! This is what happened in your homework problem

$$x'(t) = 14 x - .5 x^2 - x y$$
$$y'(t) = 16 y - .5 y^2 - x y$$

 pictures on next page...

$x' = 14x - 2x^2 - xy$
$y' = 16y - 2y^2 - xy$

9.9458, 5.0979

$x' = 14*x - .5*x^2 - x*y$
$y' = 16*y - .5*y^2 - x*y$

11.205, -4.8565

seperatrix
or
stable
orbit fn
$\begin{bmatrix} 12 \\ 8 \end{bmatrix}$

notice
how it
divides
the long
term
limits
based
on $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

Another model is the classical predator prey model, for prey $x(t)$ and predator $y(t)$. All constants are positive:

$$x'(t) = ax - pxy = x(a - py) = F(x,y) \quad \text{see warmup}$$
$$y'(t) = -by + qxy = y(-b + qx) = G(x,y)$$

Exercise 1 $\quad$ 2

a) Find the 4 equilibrium solutions
$\qquad$ constant

$$F(x,y) = 0 = x(a - py)$$
$$G(x,y) = 0 = y(-b + qx)$$

So $\quad x = 0 \quad$ or $\quad a - py = 0$

If $x = 0$ then $y = 0 \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $-b + qx = 0 \rightarrow$ inconsistent no sol'n.

If $a - py = 0 \quad y = \frac{a}{p}$
then $y = 0 \rightarrow$ inconsistent
or $-b + qx = 0 \quad x = \frac{b}{q}$

$\longleftarrow \begin{bmatrix} a/p \\ b/q \end{bmatrix}$

b) The most interesting equilibrium solution is the one in the first quadrant,

$$(x_E, y_E) = \left(\frac{b}{q}, \frac{a}{p} \right).$$

Show that the linearization at this equilibrium point always yields a stable center, which is the borderline case. So, this equilibrium is indeterminant for the nonlinear system. It turns out that for the nonlinear system, however, this first quadrant equilbrium solution *is* always a stable center. You'll explore these ideas further in homework...

$$x(t) = x_E + u(t)$$
$$y(t) = y_E + v(t)$$

$u(t), v(t)$ measure deviations from $x_E \; y_E$

$$x'(t) = ax - pxy = F$$
$$y'(t) = -by + qxy = G$$

$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \xrightarrow[\text{Sats}]{\text{approx}} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$

$$J_{@(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}$$

$J_{@} \begin{bmatrix} x_E \\ y_E \end{bmatrix}$

$$J_{@\left(\frac{b}{q}, \frac{a}{p}\right)} = \begin{bmatrix} a - p\frac{a}{p} & -p\frac{b}{q} \\ q\frac{a}{p} & -b + q\frac{b}{q} \end{bmatrix} = \begin{bmatrix} 0 & -p\frac{b}{q} \\ q\frac{a}{p} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & -p\frac{b}{q} \\ q\frac{a}{p} & -\lambda \end{vmatrix} = \lambda^2 + \frac{qa}{p} \frac{pb}{q} = \lambda^2 + ab$$

roots $\lambda = \pm i\sqrt{ab}$
for linearized, (stable) center

Here's a particular example which shows how the predator-prey system has solutions which oscillate in time. Such behavior can be observed in nature. Depending on time we may do some computations related to this example.

prey $\quad x'(t) = x - xy$
predatn $\quad y'(t) = -y + xy$

$$\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

really is a stable center

① few predators y(t);
prey x(t) increase.

② abundant prey x(t)
causes predators y(t)
to increase, and
then, prey x(t) start to
decrease

③ prey levels x(t) get so
small the predators
die off

④ cycle repeats

x´ = x−x*y
y´ = −y+x*y

predatn

prey

2.4035, 3.992

x´ = x−x*y
y´ = −y+x*y
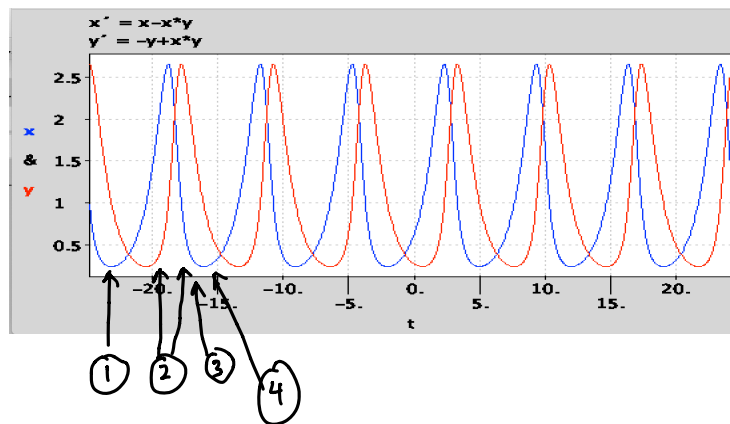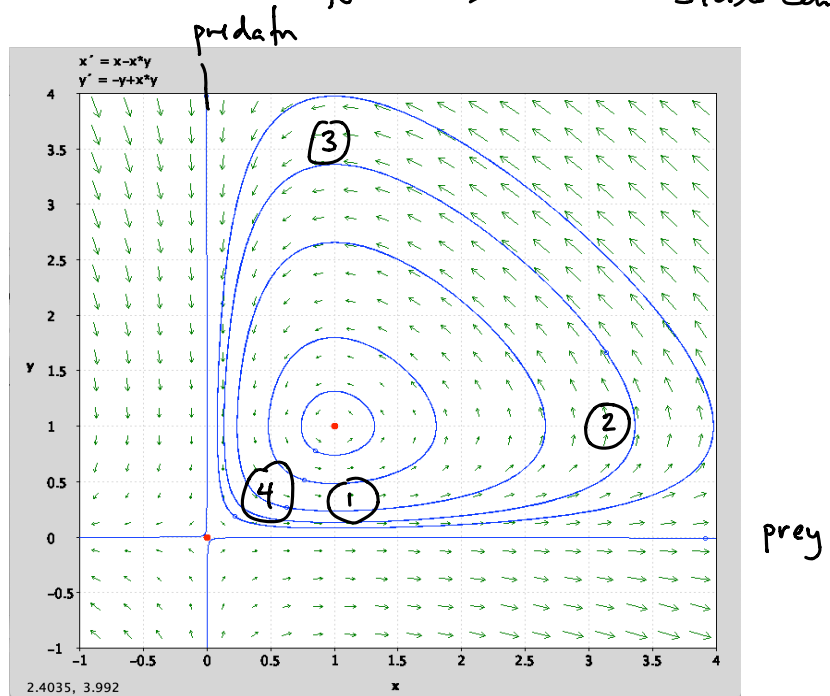
on the next page we show that
the pplane curves are actually the contour lines of
an associated conserved function

$$f(x,y) = \ln x - x + \ln y - y.$$

Since $z(t) = t - \ln t$ is concave down and has its max value of -1 at $t=1$,
$f(x,y)$ has its max value of -2 at $(x,y) = (1,1)$.
If (x_0, y_0) is close to $(1,1)$ then it will follow a contour
that also stays close to $(1,1)$. This shows $(1,1)$ is
a <u>stable center</u> for the non-linear system

These graphics from matlab are related to the phase plane on the previous page! You'll figure out the connection in your homework (see text as well), and we might discuss it in class. Matlab script:

```
%Predator-prey example
X=[.1:.1:4]
Y=[.1:.1:4]
[x,y]=meshgrid(X,Y)
z=log(x)-x +log(y)-y
%This command will plot the graph of z=f(x,y)
% as well as a contour plot below it.
figure
surfc(z)
colormap('gray')
title('predator-prey contour lines')
|
```
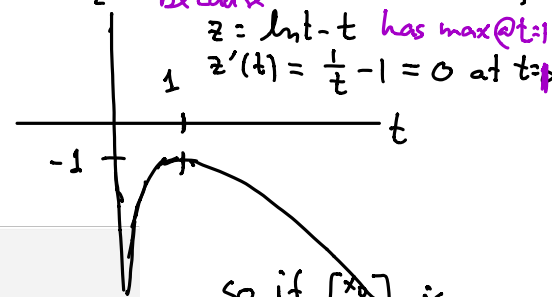
output:

predator-prey contour lines
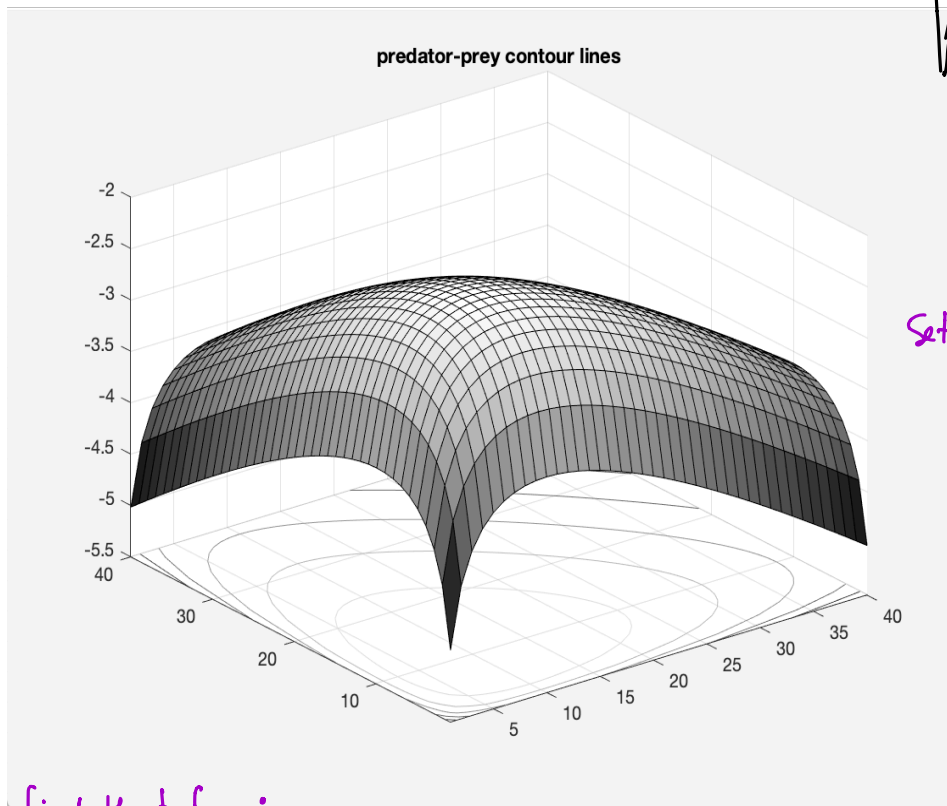
$f(x,y) = \ln x - x + \ln y - y$

has its max at $(1,1)$ because

$z = \ln t - t$ has max @ $t=1$

$z'(t) = \frac{1}{t} - 1 = 0$ at $t=1$

So if $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is close to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $f(x_0, y_0)$ is close to its max, so $[x(t), y(t)]$ stays close to $(1,1)$. Because

Set up: predator-prey curves follow contours of a function which has a strict max at equil pt, (and is "strictly" concave down)

How to find that fcn??

start here to find that fcn:

$x'(t) = x - xy$
$y'(t) = -y + xy$ so $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

$\left(\frac{d}{dt} y(x(t)) = \frac{dy}{dx} \frac{dx}{dt} \right.$

$\frac{y'(t)}{x'(t)} = \frac{dy}{dx}$

or

$(x(t), y(t))$

$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$\ln y - y = -\ln x + x + C$

$\boxed{\ln y - y + \ln x - x = C}$

$\frac{dy}{dx} = \frac{-y + xy}{x - xy}$

$\frac{dy}{dx} = \frac{y(-1+x)}{x(1-y)}$

$\frac{1-y}{y} dy = \int \frac{-1+x}{x} dx$

$\int \frac{1}{y} - 1 \, dy = \int -\frac{1}{x} + 1 \, dx$

Solution trajectories $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ follow level curves of

conclusion $f(x,y) = \ln y - y + \ln x - x$

we found that fcn!!
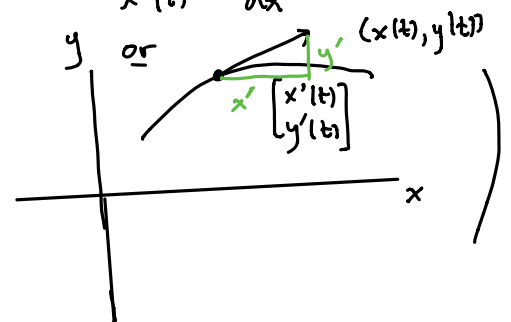
Fri Mar 22
6.4 Nonlinear mechanical systems

<u>Announcements:</u> • Pick up applications handout & completed hw assignment
                              finish   w10.1 today
                      • Wed notes first.   w10.2

<u>Warm-up Exercise:</u> we'll talk about the application handout - you could
                       look it over

Math 2280-002   Friday ~~January~~ March 22 Supplement.

We discussed a simplified linear Glucose-insulin model before break, and more recently we've been discussing nonlinear interactions between two "populations". Mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more comprehensive) models in the same vein.

Prof. Fred Adler - who has a joint appointment in Math and Biology, worked with a Math graduate student Chris Remien and collaborated with University Hospital around 7 years ago to model liver poisoning by acetominophen (brand name Tyleonol). Accidental overdoses of that particular painkiller are a leading cause of liver failure. They studied a non-linear system of 8 first order differential equations, and came up with a new diagnostic test which can be useful in deciding the extent and time frame of the overdose, in patients who are brought to the hospital unconscious. This is decisive information in whether the antidote will still save their liver, or whether they need a liver transplant to survive.

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Math Can Save Tylenol Overdose Patients

NEW WAY FOR DOCS TO PREDICT WHO NEEDS LIVER TRANSPLANTS

Tweet this post

⊞ Share   Media Contacts   *differential*

Feb. 27, 2012 – University of Utah mathematicians developed a set of calculus equations to make it easier for doctors to save Tylenol overdose patients by quickly estimating how much painkiller they took, when they consumed it and whether they will require a liver transplant to survive.

"It's an opportunity to use mathematical methods to improve medical practice and save lives," says Fred Adler, a professor of mathematics and biology and coauthor of a study that developed and tested the new method.

The study of acetaminophen – the generic pain and fever medicine sold as Tylenol and in many other nonprescription and prescription drugs – was set for publication within a week in *Hepatology*, a journal about liver function and disease.

Adler, math doctoral student Chris Remien and their colleagues showed that using only four common medical lab tests – known as AST, ALT, INR and creatinine – the equations can quickly and accurately predict which Tylenol overdose patients will survive with medical treatment and which will die unless they receive a liver transplant.

The researchers analyzed the records of 53 acetaminophen overdose patients treated at the University of Utah's University Hospital to test the equations and show they quickly and accurately predicted, in retrospect, which patients survived and which died.

Speed is essential in listing acute liver failure patients as candidates for transplant, says study coauthor Norman Sussman, a former University of Utah liver doctor now at the Baylor College of Medicine in Houston.

There's a link to their published paper in the previous link, and it would also be easy to find on google scholar. For fun, I copied and pasted the non-linear system of first order differential equations that they study. Notice the quadratic interaction terms for how the various quanitites interact. This is typical in these sorts of models, because chemical reactions typically happen at rates proportional to the amounts of reactants in the system.

Partial key:

APAP = Acetaminophen

NAPQ1 = one of the toxins

GSH = Gluthione which binds to the toxin and eliminates it

AST, ALT = enzymes released by damaged hepatocytes, which have known typical clearance time curves, and which are measured upon admission, along with the "prothrombin time/international normalized ratio (INR)" clotting factor, and then the systems is solved in the negative time direction to deduce approximate time and amount of APAP ingestion.

http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

APAP $\qquad \dfrac{dA}{dt} = -\dfrac{\alpha}{H_{max}} AH - \delta_a A$

NAPQI $\qquad \dfrac{dN}{dt} = \dfrac{qp\alpha}{H_{max}} A - \gamma NG$

GSH $\qquad \dfrac{dG}{dt} = \kappa - \gamma NG - \delta_g G$

Functional Hepatocytes $\qquad \dfrac{dH}{dt} = rH\left(1 - \dfrac{H+Z}{H_{max}}\right) - \eta NH$

Damaged Hepatocytes $\qquad \dfrac{dZ}{dt} = \eta NH - \delta_z Z$

AST $\qquad \dfrac{dS}{dt} = \dfrac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$
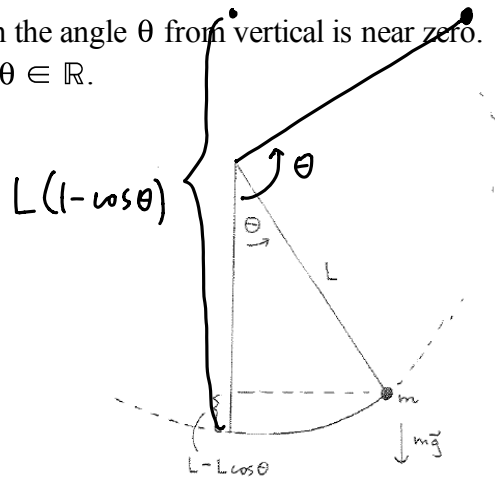
ALT $\qquad \dfrac{dL}{dt} = \dfrac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$

Clotting Factor $\qquad \dfrac{dF}{dt} = \beta_f \left(\dfrac{H}{H_{max}} - F\right)$

<u>Example 1)</u> The rigid rod pendulum.

We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.

$L(1-\cos\theta)$

$L - L\cos\theta$

In Chapter 3 we used conservation of energy to derive the dynamics for this (now) swinging, or possibly rotating, pendulum. There were no assumptions about the values of θ in that derivation of the non-linear DE (it was only when we linearized that we assumed θ was near zero). We began with the total energy

$$TE = KE + PE = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m\left(L\,\theta'\,(t)\right)^2 + m\,g\,L\left(1 - \cos\left(\theta(t)\right)\right)$$

And set $TE'\,(t) \equiv 0$ to arrive at the differential equation

$$\theta''\,(t) + \frac{g}{L}\,\sin\left(\theta(t)\right) = 0 \quad.$$

$$\Theta'' + \frac{g}{L} \sin\Theta = 0 \qquad \frac{d}{dt}\begin{bmatrix} \Theta \\ \Theta' \end{bmatrix} = \begin{bmatrix} \Theta' \\ -\frac{g}{L}\sin\Theta \end{bmatrix}$$

We see that the constant solutions $\theta(t) = \theta_*$ must satisfy $\sin(\theta_*) = 0$, i.e. $\theta_* = n\pi$, $\pi \in \mathbb{R}$. In other words, the mass can be at rest at the lowest possible point (if θ is an even multiple of π), but also at the highest possible point (if θ is any odd multiple of π). We expect the lowest point configuration to be a "stable" constant solution, and the other one to be "unstable".

We will study these stability questions systematically using the equivalent first order system for
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$
when $\theta(t)$ represents solutions the pendulum problem. You can quickly check that this is the system
$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x).$$
Notice that constant solutions of this system, $x' \equiv 0$, $y' \equiv 0$, equivalently
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_* \\ y_* \end{bmatrix} \quad \text{equals constant}$$
must satisfy $y_* = 0$, $\sin(x_*) = 0$, In other words, $x = n\pi$, $y = 0$ are the equilibrium solutions. These correspond to the constant solutions of the second order pendulum differential equation, $\theta = n\pi$, $\theta' = 0$.

Here's a phase portrait for the first order pendulum system, with $\frac{g}{L} = 1$, see below.

a) Locate the equilibrium points on the picture and verify algebraically.

b) Interpret the solution trajectories in terms of pendulum motion.

c) Looking near each equilibrium point, and recalling our classifications of the origin for linear homogenous systems (spiral source, spiral sink, nodal source, nodal sink, saddle, stable center), how would you classify these equilbrium points and characterize their stability?

equil soltns were

$\quad X = n\pi \quad n \in \mathbb{Z} \quad (x(t) = \theta(t))$
$\quad y = 0 \qquad\qquad\qquad (y(t) = \theta'(t))$

$X = n\pi \quad$ n even
$\quad$ looks like center

$X = n\pi \quad$ n odd
$\quad$ saddle

▽ really <u>are</u> stable centers

θ'

x′ = y
y′ = -sin(x)

$(-\pi, 0)$ $(0,0)$ $(\pi,0)$ $(2\pi,0)$
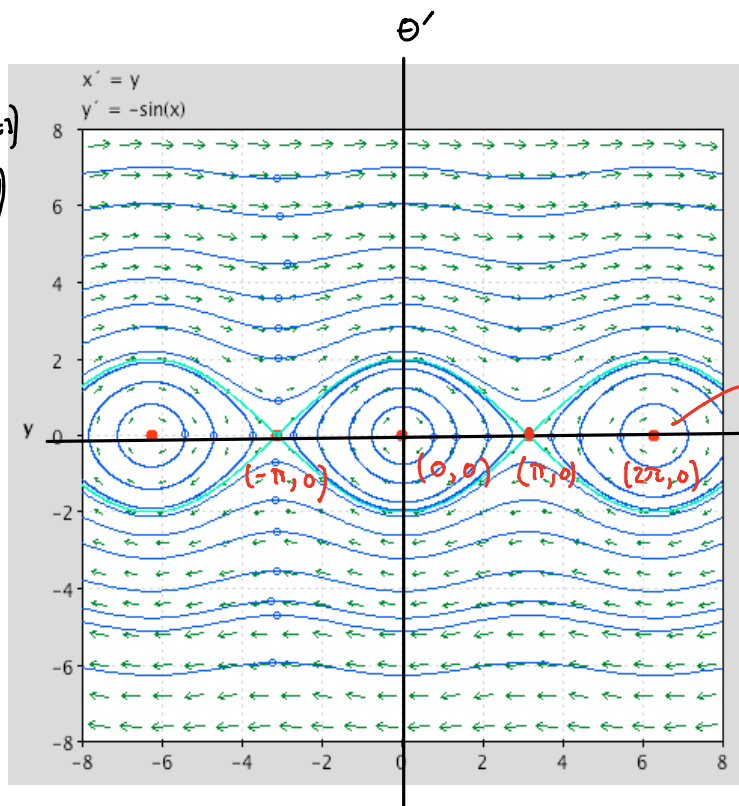
around &
around
pos θ dir.

back &
θ forth.

around &
around in
neg θ dir

$TE = \frac{1}{2}mv^2 + mgL(1-\cos\theta) \longrightarrow TE(x,y) = \frac{1}{2}mL^2 y^2 + mgL(1-\cos x)$

is constant fn solution trajectories

$x = \theta(t)$
$y = \theta'(t)$
$v = L\theta'(t)$

So for $\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} 2\pi n \\ 0 \end{bmatrix}$

$TE = 0$,
smallest possible

If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is close to

$\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ its TE is close to 0

so $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ stays close to $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$

what y value
makes PE
as small as possible
$y = 0$

what x
values make
this as small
as possible
$x = 2\pi n$,
$n \in \mathbb{Z}$

stable
center

Exercise 1 Work out the Jacobian matrices and linearizations at the equilibria for the undamped rigid rod pendulum system on the previous page
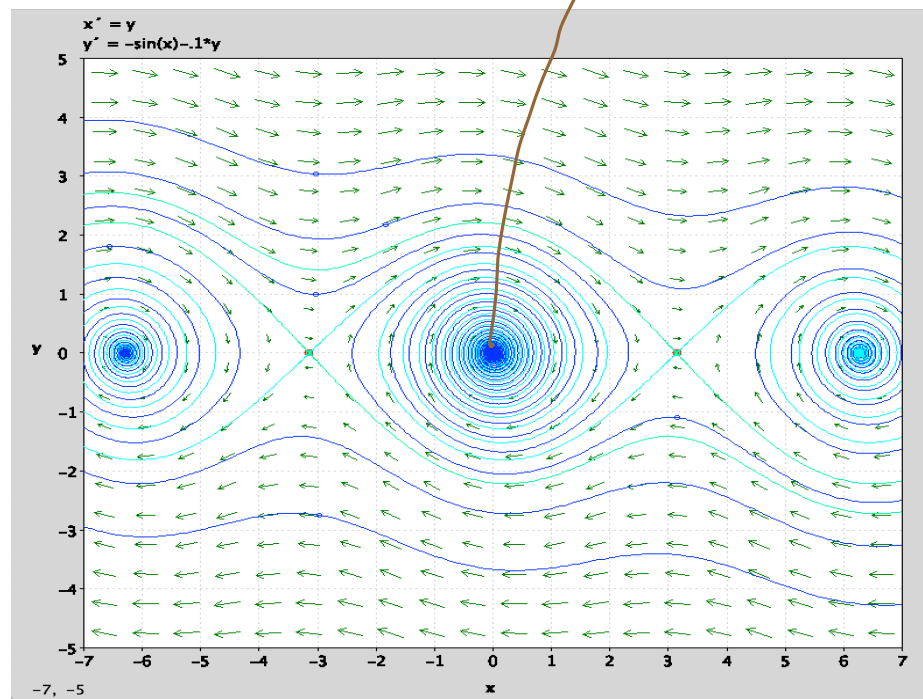
$$x'(t) = y$$

$$y'(t) = -\frac{g}{L}\sin(x)$$

Exercise 2 What happens when you add damping?

$$\theta''(t) + c\,\theta'(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x) - c\,y$$

spiral sinks

C = .1

x˙ = y
y˙ = −sin(x)−.1*y

−7, −5

C = 3
overdamped

x˙ = y
y˙ = −sin(x)−3*y

−6.939, 3.4043