Math 2280-002
Week 10, March 18-22 5.3, 6.1-6.4
Mon Mar 18
5.3 - phase portraits for homogeneous systems of two linear DE's - summary of complex eigendata case from Friday, and discussion of real eigendata examples; 6.1 Introduction to systems of two autonomous first order differential equations.

Announcements:

Warm-up Exercise:

5.3 phase portraits for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A. This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points. On Friday before break we analyzed the case of complex eigendata. Today we'll analyze what happens with real eigendata.

Here's a summary of what happens if the matrix A (with real entries) has complex eigendata:

<u>complex eigenvalues</u> (from Friday before break, included here for completeness): Let $A_{2\times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i$. Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the <u>complex</u> solution

$$
\begin{aligned}
\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\
&= e^{p t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\
&= \left[e^{p t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p t}\sin(q\,t)\underline{\boldsymbol{b}}\right]\\
&\quad + i\left[e^{p t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p t}\cos(q\,t)\underline{\boldsymbol{b}}\right] .
\end{aligned}
$$

Thus, the general <u>real</u> solution is a linear combination of the real and imaginary parts of the solution above. I put the linear combination weights c_1, c_2 on the right instead of the left in the expression below, to facilitate seeing the matrix factorization in the next step:

$$
\begin{aligned}
\underline{\boldsymbol{x}}(t) = e^{p t}&[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}]\,c_1\\
&+ e^{p t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}]\,c_2 .
\end{aligned}
$$

We can rewrite $\underline{\boldsymbol{x}}(t)$ as

$$
\underline{\boldsymbol{x}}(t) = e^{p t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

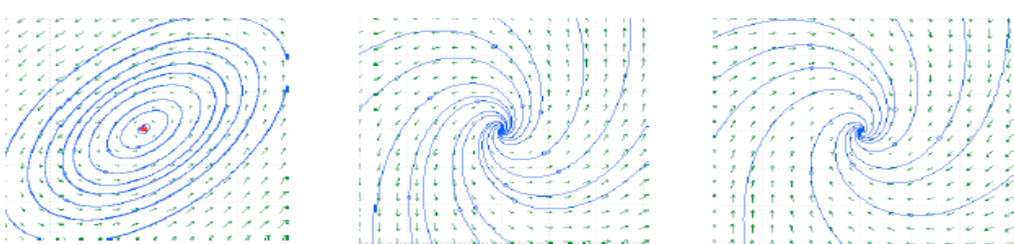
$$
\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

$$
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

Thus $\underline{\boldsymbol{x}}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center
Re(λ)=0

spiral source
Re(λ)>0

spiral sink
Re(λ)<0

Real eigenvalues If the matrix $A_{2 \times 2}$ is diagonalizable, i.e. if there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of A), then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ is

$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2$$

- And, for each $\underline{\boldsymbol{x}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2$ the value of the tangent field at x is

$$A\,\underline{\boldsymbol{x}} = A\left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2\right) = c_1\lambda_1\underline{\boldsymbol{v}}_1 + c_2\lambda_2\underline{\boldsymbol{v}}_2 .$$

(The text discusses the case of non-diaganalizable A. This can only happen if $det(A - \lambda I) = \left(\lambda - \lambda_1\right)^2$, but the $\lambda = \lambda_1$ eigenspace is defective so that its dimension is one instead of two.)

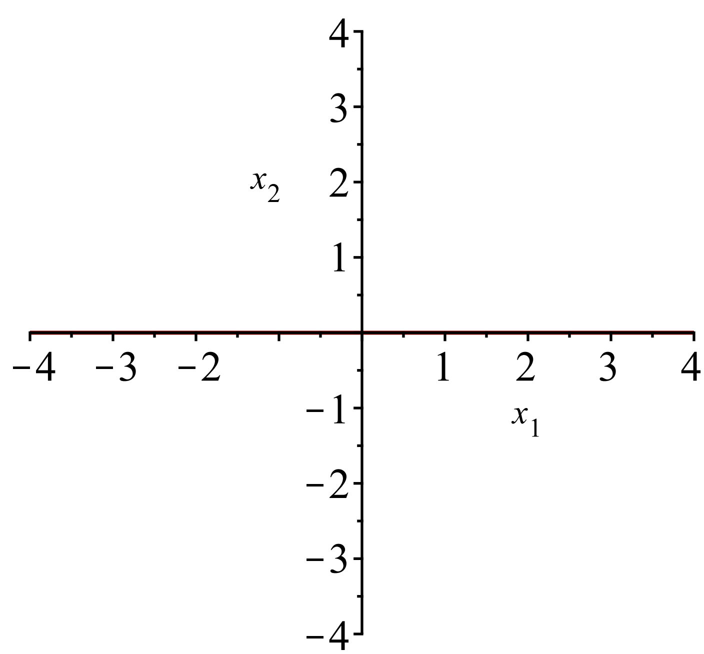
Exercise 1) Here is an example of what happens when A has two real eigenvalues of opposite sign. Consider the system

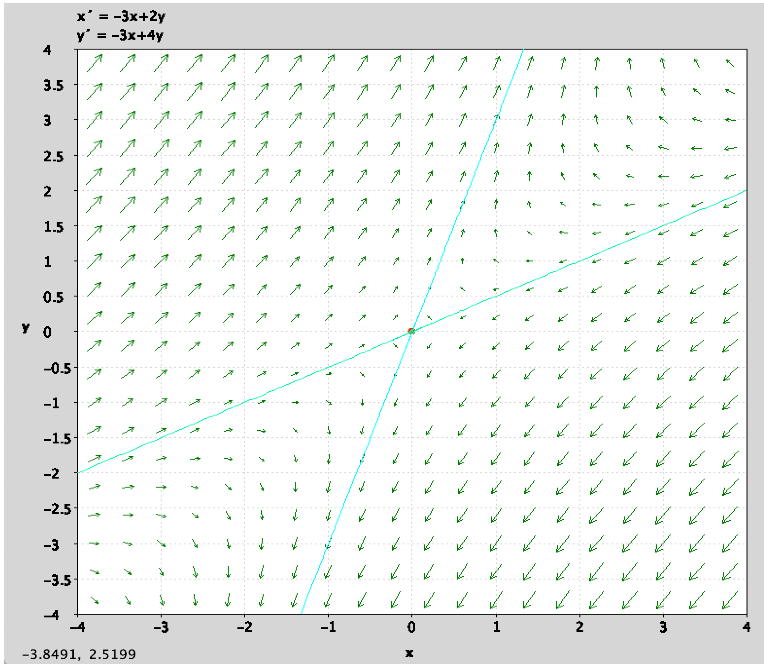
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

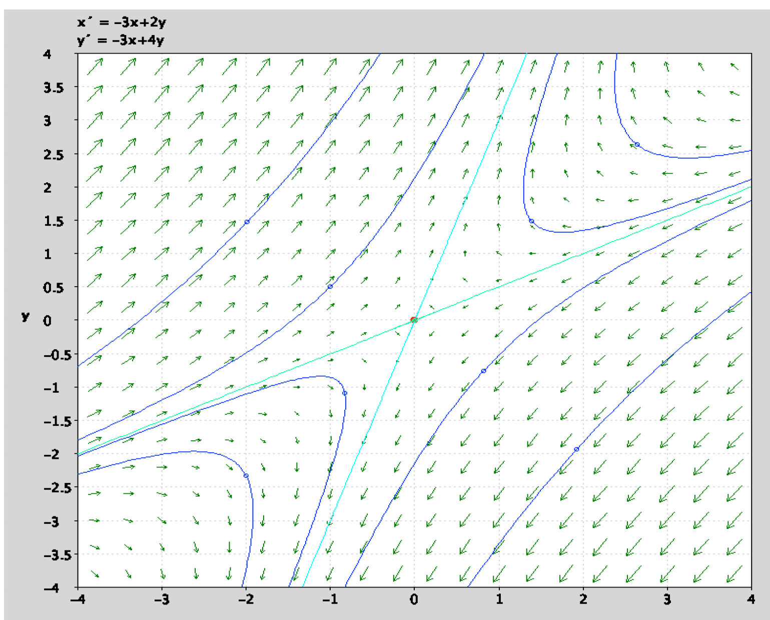
a) The eigendata for A is

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}, \qquad E_{\lambda=3} = span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

b) Use just the eigendata to sketch the tangent vector field on the plot below. Begin by sketching the two eigenspaces.
c) Use the general solutions to the DE system to overlay representative solution curves. Notice that your work in b, c is consistent.




x´ = −3x+2y
y´ = −3x+4y
y
x
−3.8491, 2.5199


x´ = −3x+2y
y´ = −3x+4y
y
x
−3.9303, −2.0606

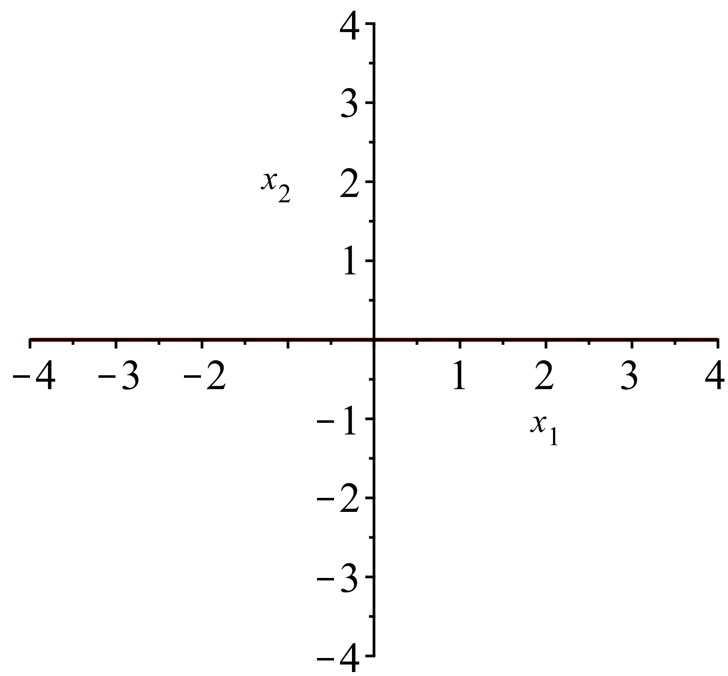
Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

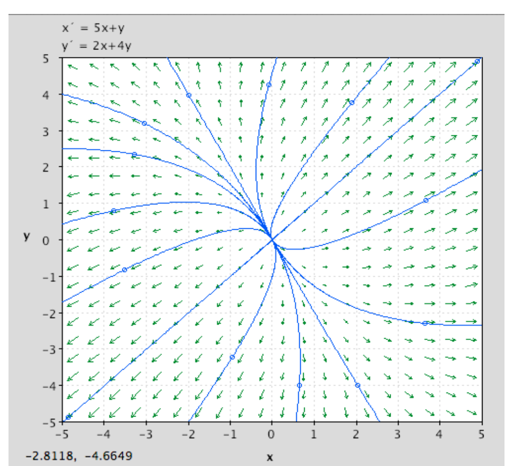
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Use the eigendata of A to find the general solution to the first order system of DE's.

$$E_{\lambda=3} = span\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\},\ E_{\lambda=6} = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves. First sketch the eigenspaces.





Theorem: Time reversal: If $\underline{\mathbf{x}}(t)$ solves

$$\underline{\mathbf{x}}' = A\,\underline{\mathbf{x}}$$

then $\underline{\mathbf{z}}(t) := \underline{\mathbf{x}}(-t)$ solves

$$\underline{\mathbf{z}}' = (-A)\underline{\mathbf{z}}$$

proof: by the chain rule,

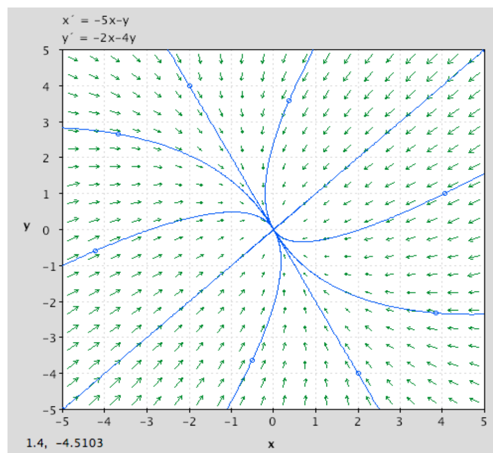
$$\underline{\mathbf{z}}'(t) = \underline{\mathbf{x}}'(-t)\cdot(-1) = -\underline{\mathbf{x}}'(-t) = -A\,\underline{\mathbf{x}}(-t) = -A\,\underline{\mathbf{z}}.$$

Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A?

b) Describe how the eigendata of the matrix in the system below is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits and solutions are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



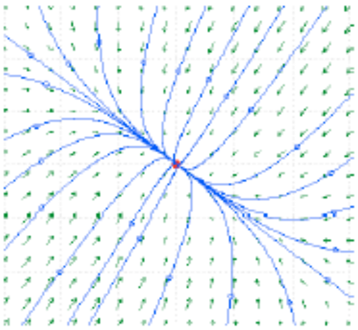
summary: In case the matrix $A_{2 \times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

has general solution

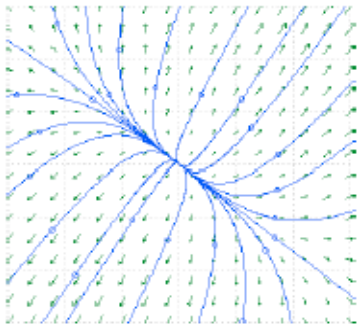
$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 \ .$$

If each eigenvalue is non-zero, the three possibilities are:



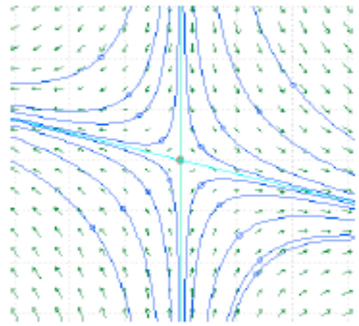
nodal sink

$\lambda_1, \lambda_2 < 0$



nodal source

$\lambda_1, \lambda_2 > 0$



saddle point

$\lambda_1 < 0 < \lambda_2$

Introduction to Chapter 6. (We'll return to Chapter 5 after we discuss Chapter 6.) This chapter is about general (non-linear) systems of two first order differential equations for $x(t), y(t)$, i.e.

$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$

which we often abbreviate, by writing

$$x' = F(x, y, t)$$
$$y' = G(x, y, t) \ .$$

If we assume further that the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t, i. e.

$$x' = F(x, y)$$
$$y' = G(x, y)$$

then we call such a system autonomous. Autonomous systems of first order DEs are the focus of most of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2.

Constant solutions to an autonomous differential equation or system of DEs are called equilibrium solutions. Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system

$$F(x, y) = 0$$
$$G(x, y) = 0$$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems

$$x' = F(x, y)$$
$$y' = G(x, y)$$

are called stable if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.

- Equilibrium solutions are unstable if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called asymptotically stable if they are stable and furthermore,

IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \rightarrow \infty$.

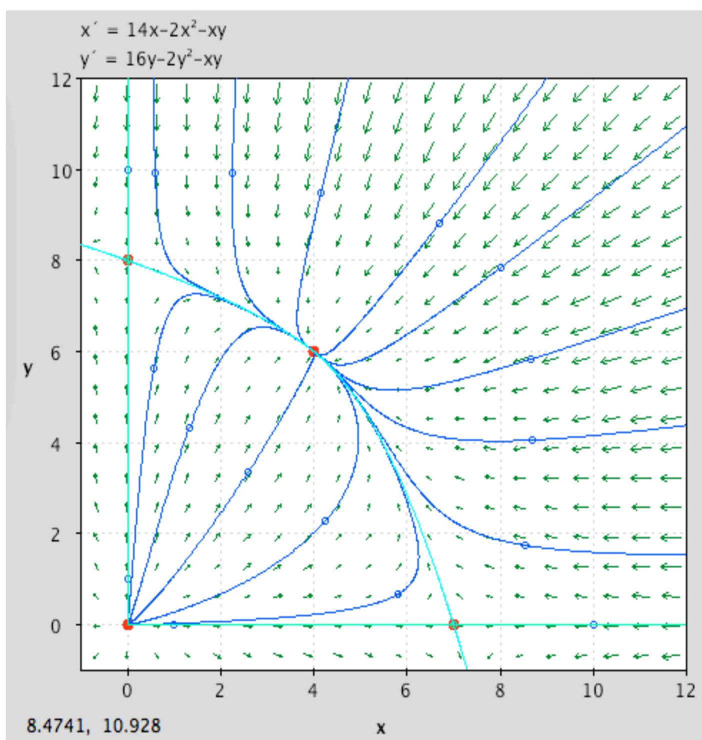
(Notice these definitions are completely analogous to our discussion in Chapter 2.)

Exercise 4) Consider the "competing species" model from section 6.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy.$$

a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other). Hint: to understand why this model is plausible for $x(t)$ consider the normalized birth rate rate $\frac{x'(t)}{x(t)}$, as we did in Chapter 2.

b) Find the four equilibrium solutions to this competition model, algebraically.
c) What does the phase portrait below indicate about the dynamics of this system?
d) Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability, based on what the phase portrait looks like near each equilibrium solution?



Tues Mar 19
6.1-6.2 Autonomous systems of two first order differential equations; linearization near equilibrium solutions.

Announcements:

Warm-up Exercise:

Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math 2280 course. (The "linear drag" velocity model, Newton's law of cooling, small oscillation pendulum motion, and the damped spring equation were all linearizations.) It's important to understand how to linearize in general, because linearized differential equations can often be used to understand stability and solution behavior near equilibrium points, for the original differential equations. Today we'll talk about linearizing *systems* of DE's, which we've not done before in this course.

An easy case of linearization in Exercise 4 from Monday's notes is near the equilbrium solution $[x_*, y_*]^T = [0, 0]^T$. It's pretty clear that our rabbit-squirrel population system

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

linearizes to

$$x'(t) = 14x$$
$$y'(t) = 16y$$

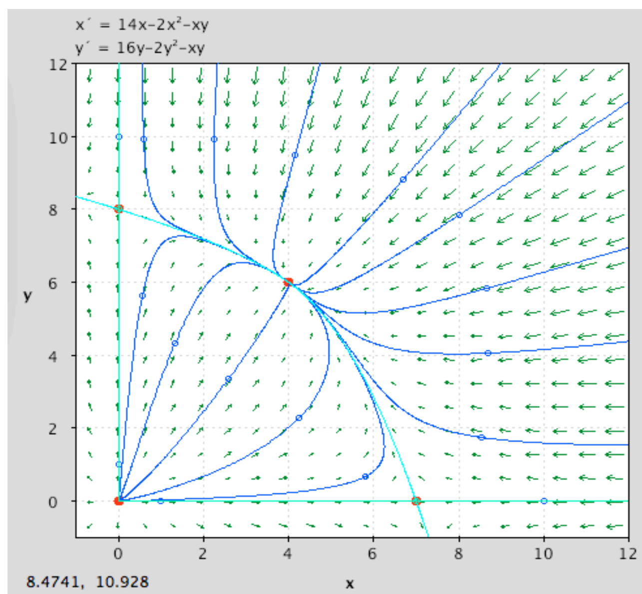
i.e.

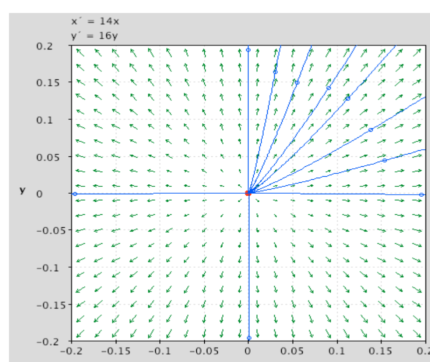
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:





We use multivariable Calculus to linearize at equilibrium points that are not the origin. (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter.) Here's how: Consider the autonomous system

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \quad .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \; .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \; .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*)\, u + \frac{\partial F}{\partial y}(x_*, y_*)\, v + \varepsilon_1(u, v)$$
$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*)\, u + \frac{\partial G}{\partial y}(x_*, y_*)\, v + \varepsilon_2(u, v)$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

The matrix of partial derivatives is called the <u>Jacobian matrix</u> for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Space to review the multivariable tangent approximation, which is *the* key concept behind multivariable differential calculus, as well as being important for change of variables in multiple variable integration.

Exercise 1) We will linearize the rabbit-squirrel (competition) model of the running example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y$$

1a) Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

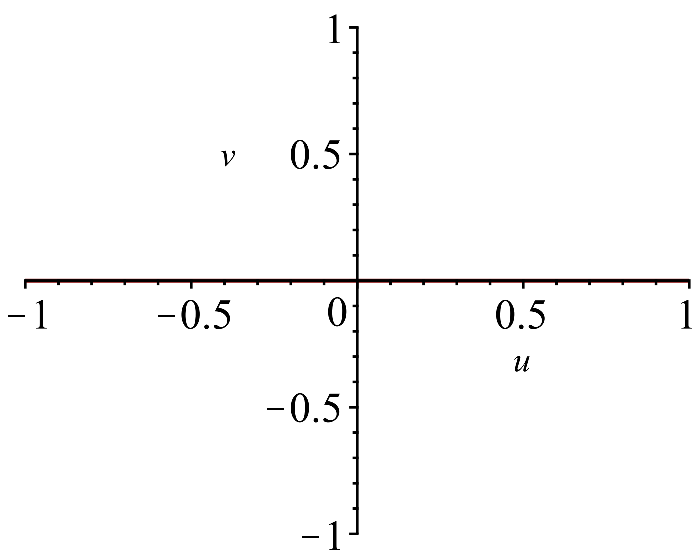
So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

1b) The matrix in the linear system of DE's above has approximate eigendata:

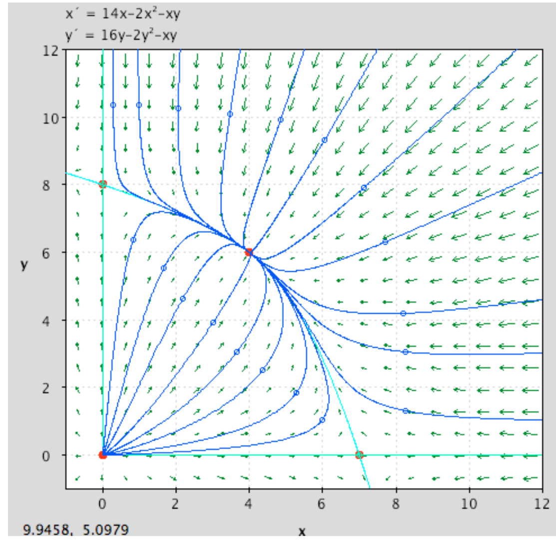
$$\lambda_1 \approx -4.7, \quad \underline{\mathbf{v}}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{\mathbf{v}}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, the phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$.



Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

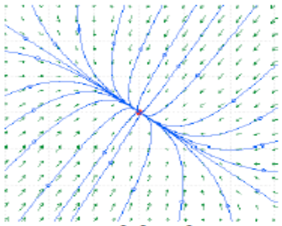
The non-linear problem and representative solution curves:



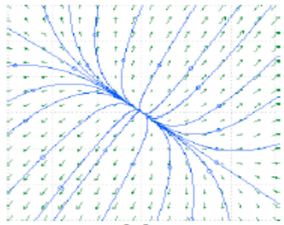
pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

```
Equilbrium Point:
    There is a nodal sink at (4, 6)
    Jacobian:
            -8          -4
            -6          -12
    The eigenvalues and eigenvectors are:
            -4.7085    (0.77218, -0.63541)
            -15.292    (0.48097, 0.87674)
```

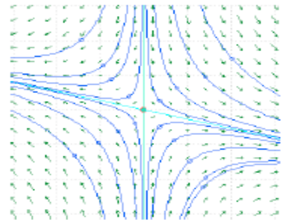
The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



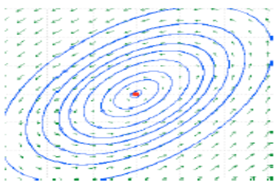
nodal sink
$\lambda_1, \lambda_2 < 0$



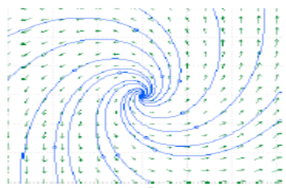
nodal source
$\lambda_1, \lambda_2 > 0$



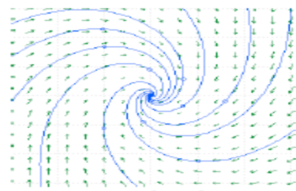
saddle point
$\lambda_1 < 0 < \lambda_2$



center
$Re(\lambda)=0$



spiral source
$Re(\lambda)>0$



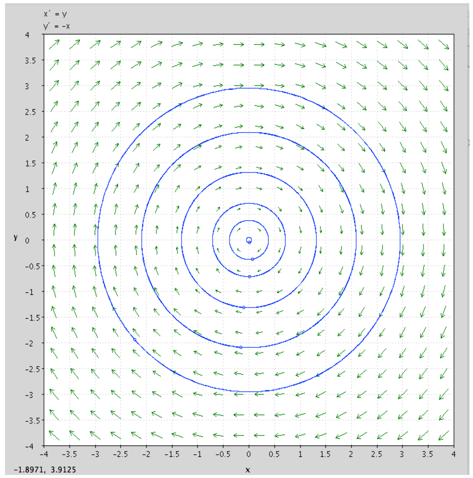
spiral sink
$Re(\lambda)<0$

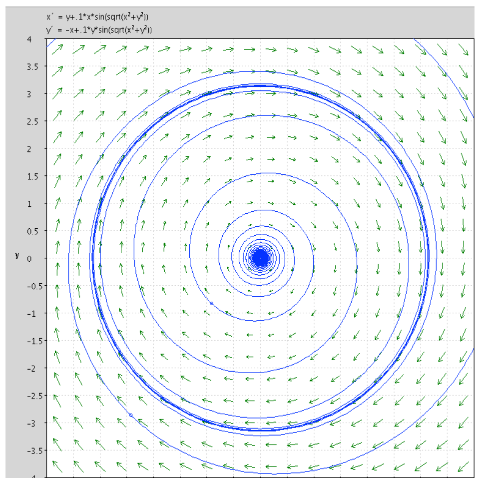
Theorem: Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.

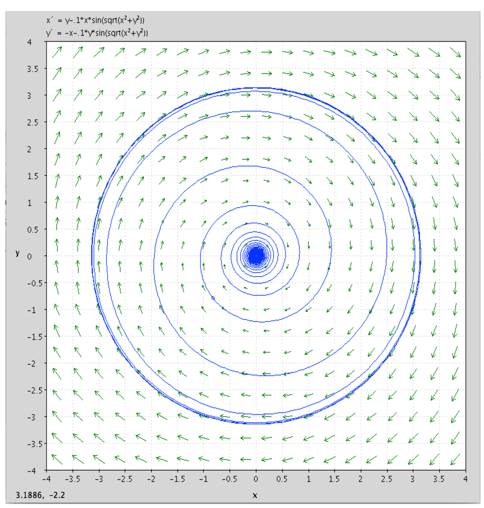
(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.

(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Fun examples of borderline cases where the linearization at the origin has purely imaginary eigenvalues, so the origin is a stable center for the linearization but all three flavors for the three nonlinear systems:







Wed Mar 20
6.3 Ecological models, continued

Announcements:

Warm-up Exercise:

There are many interesting two-species models. In class and in one of your homework problems we've considered examples of the *logistic competition model* between two species:

$$x'(t) = a_1 x - b_1 x^2 - c_1 x y$$
$$y'(t) = a_1 y - b_2 y^2 - c_2 x y$$

Here the constants $a_1, a_2, b_1, b_2, c_1, c_2$ are all positive. It turns out that if the logistic inhibition, as measured by the product $b_1 b_2$ is stronger than the competitive pressure as measured by $c_1 c_2$, i.e.

$$b_1 b_2 > c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always asymptotically stable. This is what happened in our class example

$$x'(t) = 14 x - 2 x^2 - x y$$
$$y'(t) = 16 y - 2 y^2 - x y .$$

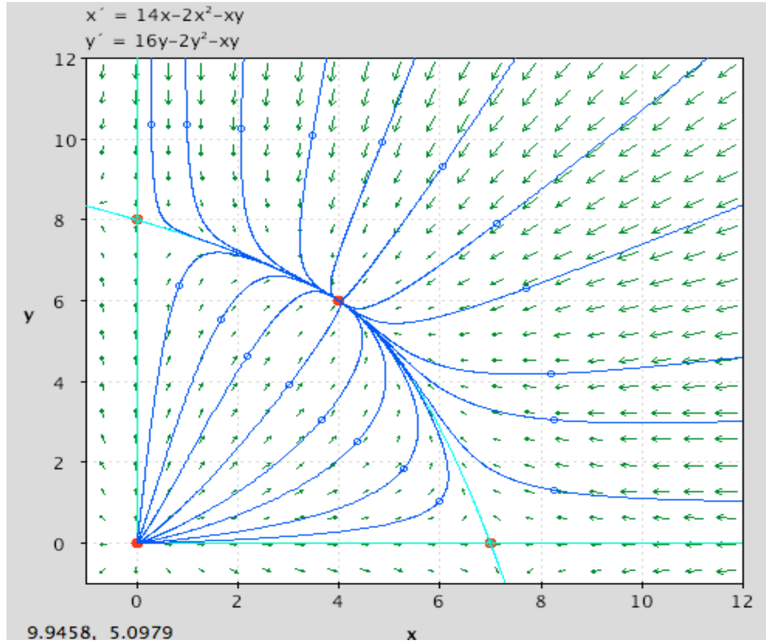
On the other hand, if

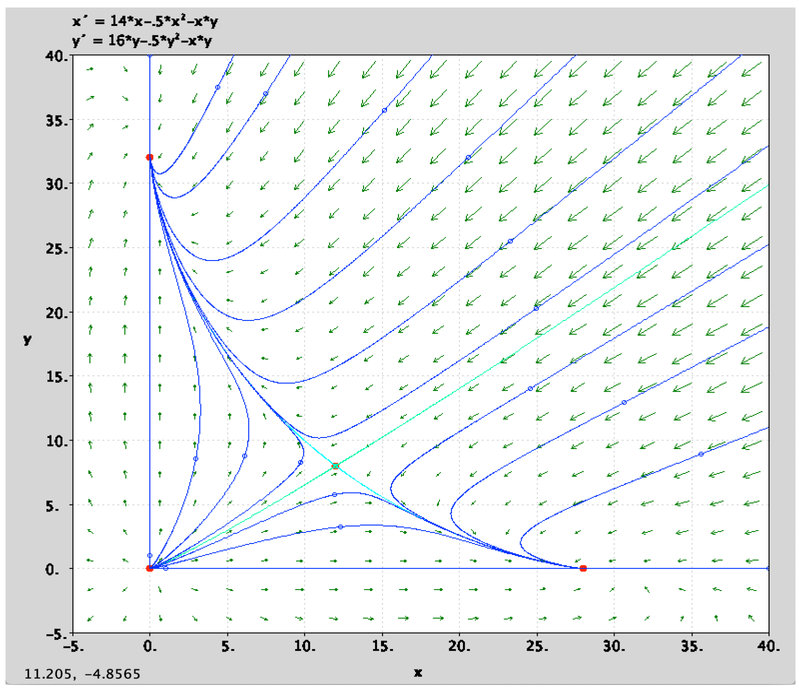
$$b_1 b_2 < c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always unstable! This is what happened in your homework problem

$$x'(t) = 14 x - .5 x^2 - x y$$
$$y'(t) = 16 y - .5 y^2 - x y$$

pictures on next page...


x´ = 14x−2x²−xy
y´ = 16y−2y²−xy
y
x
9.9458, 5.0979


x´ = 14*x−.5*x²−x*y
y´ = 16*y−.5*y²−x*y
y
x
11.205, −4.8565

Another model is the classical predator prey model, for prey $x(t)$ and predator $y(t)$. All constants are positive:

$$x'(t) = a\,x - p\,x\,y = x\,(a - p\,y)$$
$$y'(t) = -b\,y + q\,x\,y = y(-b + q\,x)\ .$$

Exercise 1

a) Find the 4 equilibrium solutions

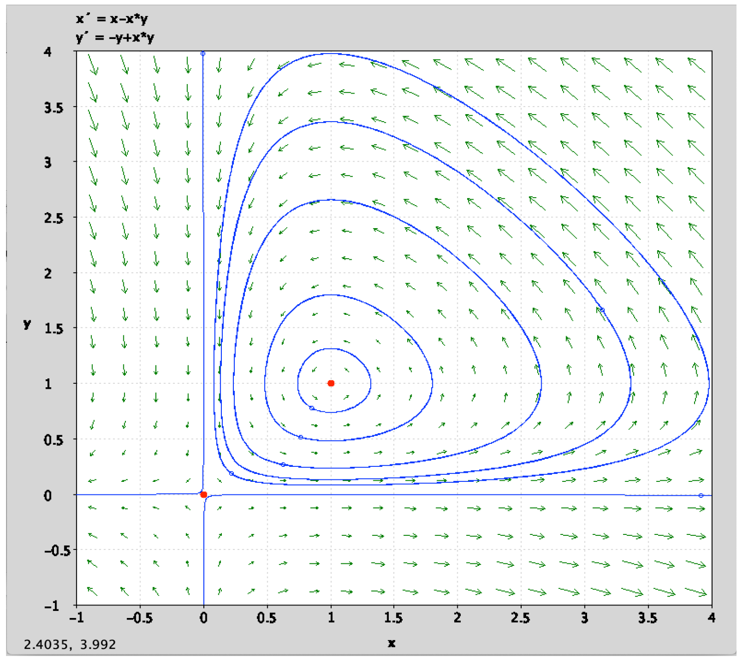
b) The most interesting equilibrium solution is the one in the first quadrant,

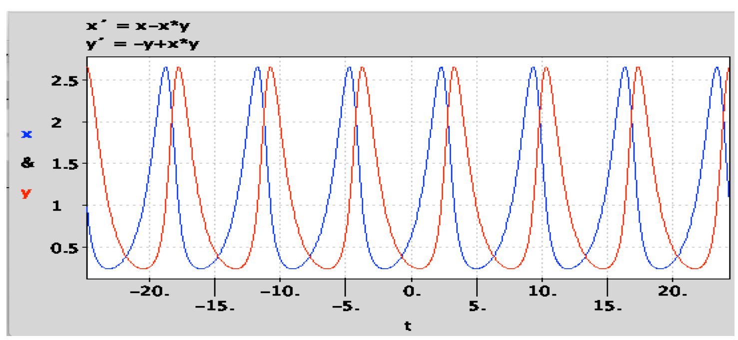
$$\left(x_E, y_E\right) = \left(\frac{b}{q}, \frac{a}{p}\right).$$

Show that the linearization at this equilibrium point always yields a stable center, which is the borderline case. So, this equilibrium is indeterminant for the nonlinear system. It turns out that for the nonlinear system, however, this first quadrant equilbrium solution *is* always a stable center. You'll explore these ideas further in homework...

Here's a particular example which shows how the predator-prey system has solutions which oscillate in time. Such behavior can be observed in nature. Depending on time we may do some computations related to this example.

$$x'(t) = x - xy$$
$$y'(t) = -y + xy$$

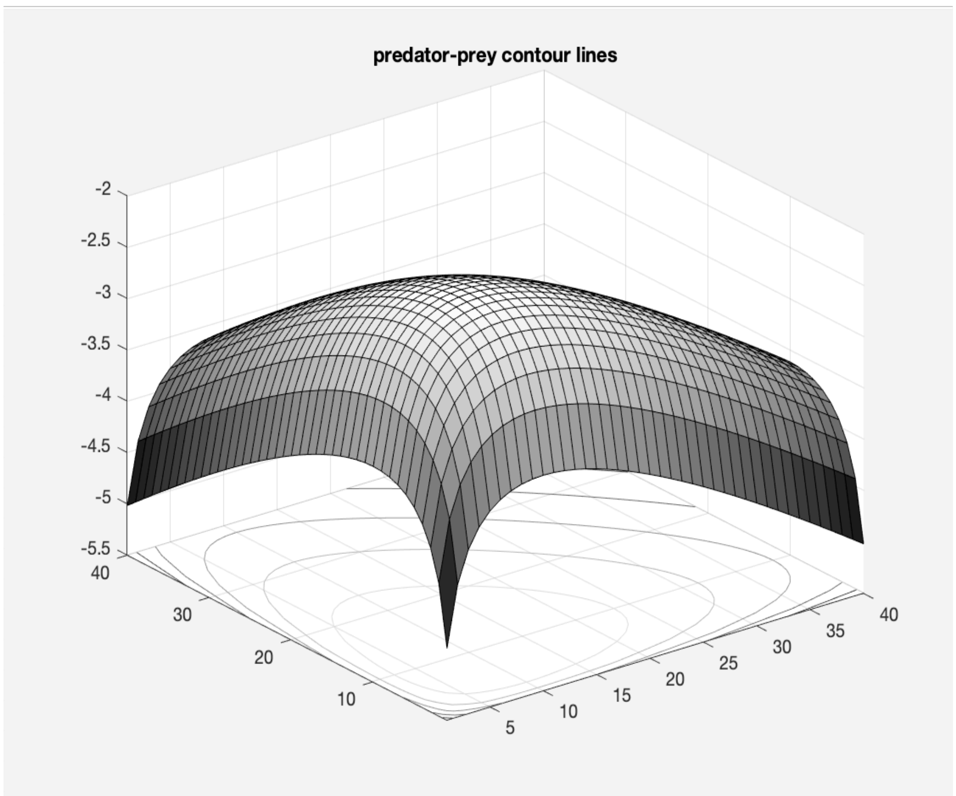




These graphics from matlab are related to the phase plane on the previous page! You'll figure out the connection in your homework (see text as well), and we might discuss it in class. Matlab script:

```
%Predator-prey example
X=[.1:.1:4]
Y=[.1:.1:4]
[x,y]=meshgrid(X,Y)
z=log(x)-x +log(y)-y
%This command will plot the graph of z=f(x,y)
% as well as a contour plot below it.
figure
surfc(z)
colormap('gray')
title('predator-prey contour lines')
```

output:



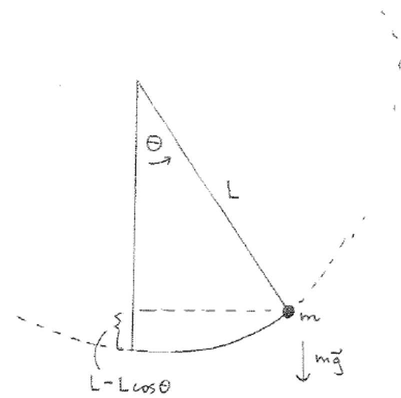
Fri Mar 22
6.4 Nonlinear mechanical systems

Announcements:

Warm-up Exercise:

Example 1) The rigid rod pendulum.

We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.



In Chapter 3 we used conservation of energy to derive the dynamics for this (now) swinging, or possibly rotating, pendulum. There were no assumptions about the values of θ in that derivation of the non-linear DE (it was only when we linearized that we assumed θ was near zero). We began with the total energy

$$TE = KE + PE = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m\left(L\,\theta'(t)\right)^2 + m\,g\,L\left(1 - \cos\left(\theta(t)\right)\right)$$

And set $TE'(t) \equiv 0$ to arrive at the differential equation

$$\theta''(t) + \frac{g}{L}\sin\left(\theta(t)\right) = 0 \ .$$

We see that the constant solutions $\theta(t) = \theta_*$ must satisfy $\sin(\theta_*) = 0$, i.e. $\theta_* = n\,\pi,\ \pi \in \mathbb{R}$. In other words, the mass can be at rest at the lowest possible point (if θ is an even multiple of π), but also at the highest possible point (if θ is any odd multiple of π). We expect the lowest point configuration to be a "stable" constant solution, and the other one to be "unstable".

We will study these stability questions systematically using the equivalent first order system for

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$

when $\theta(t)$ represents solutions the pendulum problem. You can quickly check that this is the system

$$x'(t) = y$$

$$y'(t) = -\frac{g}{L}\sin(x).$$

Notice that constant solutions of this system, $x' \equiv 0, y' \equiv 0$, equivalently

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_* \\ y_* \end{bmatrix} \text{ equals constant}$$

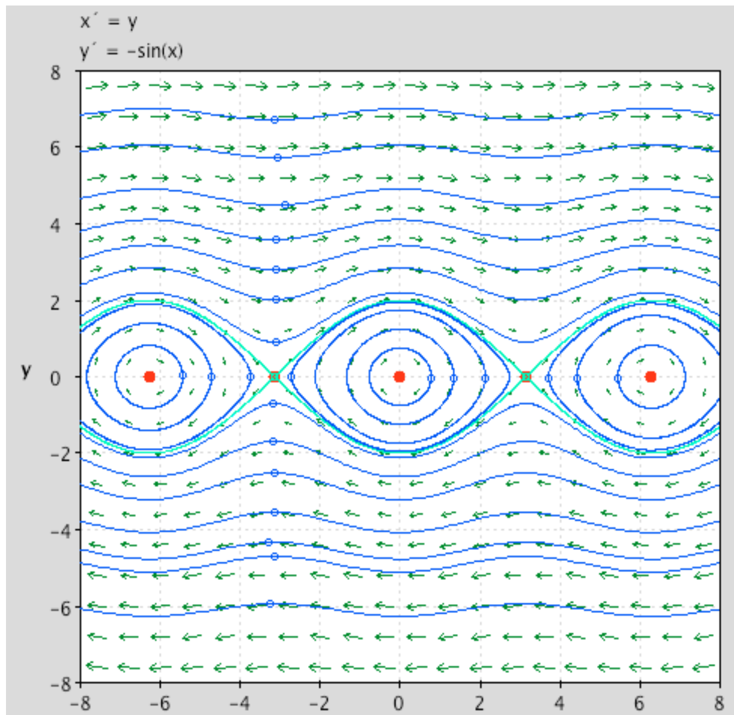
must satisfy $y_* = 0,\ \sin(x_*) = 0$, In other words, $x = n\,\pi,\ y = 0$ are the equilibrium solutions. These correspond to the constant solutions of the second order pendulum differential equation, $\theta = n\,\pi,\ \theta' = 0$.

Here's a phase portrait for the first order pendulum system, with $\frac{g}{L} = 1$, see below.

a) Locate the equilibrium points on the picture and verify algebraically.

b) Interpret the solution trajectories in terms of pendulum motion.

c) Looking near each equilibrium point, and recalling our classifications of the origin for linear homogenous systems (spiral source, spiral sink, nodal source, nodal sink, saddle, stable center), how would you classify these equilbrium points and characterize their stability?



Exercise 1 Work out the Jacobian matrices and linearizations at the equilibria for the undamped rigid rod pendulum system on the previous page

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x)$$

Exercise 2 What happens when you add damping?

$$\theta''(t) + c\,\theta'(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x) - c\,y$$