Math 2280-002 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes are for sections 1.1-1.3, and part of 1.4.

Monday January 7

- Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2280spring19

- Notice that there our first homework assignment is due next Wednesday, but that we will already have a quiz this Wednesday.

Then, let's begin!

Section 1.1 Introduction to differential equations

Definition An n^{th} *order differential equation* (DE) is any equation expressed in terms of an uspecified function $y = y(x)$ and its derivatives, for which the highest derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which after rearrangement can be written as

$$F\left(x, y(x), y'(x), y''(x),\ldots y^{(n)}(x)\right) = 0.$$

shorthand convention:

$$F\left(x, y, y', y'', y^{(n)}\right) = 0$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $(y''(x))^2 + \sin(y(x)) = 0.$

b) For $x = x(t)$, $x'(t) = 3\,x(t)\,(10 - x(t))$.

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$.

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$.

e) For $y = y(x)$, $y' = y^2$.

Related definitions:

- A specified function or functions $y(x)$ *solve(s)* the *differential equation*
$$F\left(x, y, y', y'', \; y^{(n)}\right) = 0$$

on some interval I of x-values (or is a *solution function* for the differential equation) means that $y(x)$ makes the differential equation a true identity for all x in I.

Chapters 1-2 are about first order differential equations, algebraic and graphical representations of their solutions, and applications. For first order differential equations
$$F(x, y, y') = 0$$

we can often use algebra to solve for y' in order to get what we call the **standard form** for the first order DE:

$$y' = f(x, y) \ .$$

- If we want our solution function to a first order DE to also satisfy $y\left(x_0\right) = y_0$, and if our DE is written in standard form, then we say that we are studying an *initial value problem (IVP):*
$$y' = f(x, y)$$
$$y\left(x_0\right) = y_0 \ .$$

If we can find a solution function $y(x)$ to the DE satisfying the *initial condition* $y\left(x_0\right) = y_0$, then we say that $y(x)$ *solves* the initial value problem.

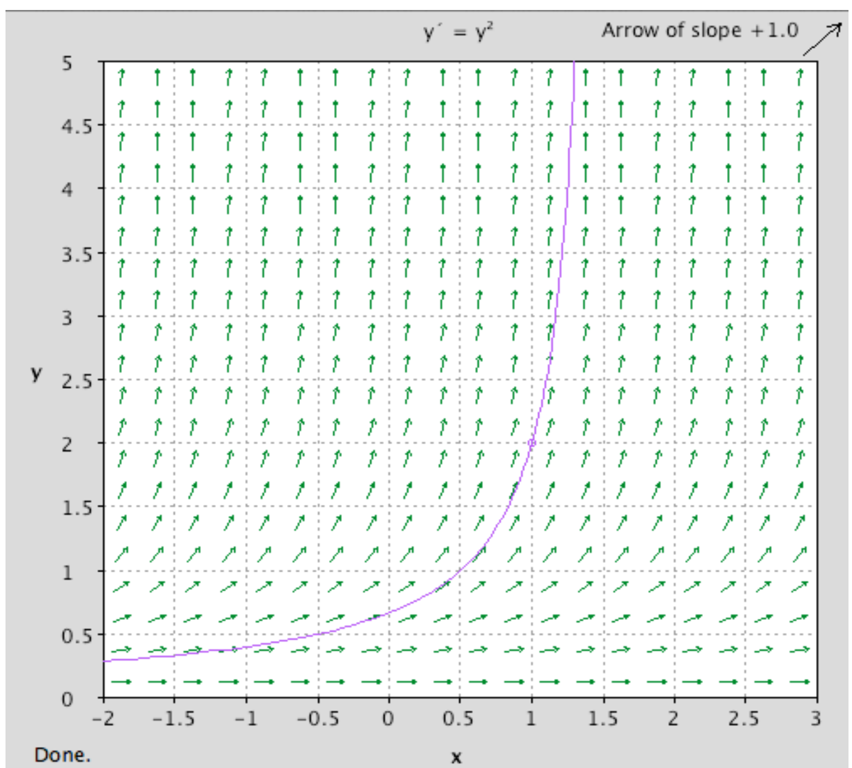
Exercise 2: Consider the differential equation $\frac{dy}{dx} = y^2$ from (1e).

a) Show that functions $y(x) = \dfrac{1}{C - x}$ *solve* the DE (on any interval not containing the constant C). We'll see how we found these functions in part 2e, but we don't need that information to check whether or not they actually solve the differential equation.

b) Find the appropriate value of C to *solve the initial value problem*
$$y' = y^2$$
$$y(1) = 2 \ .$$

2c) What is the largest interval on which your solution to 2b is defined as a differentiable function? Why?

2d) Do you expect that there are any other solutions to the IVP in 2b (on the same interval)? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below, where the line segment slopes at points (x, y) have values y^2 (because solution graphs to our differential equation will have those slopes, according to the differential equation). This might give you some intuition about whether you expect more than one solution to the IVP.



2e) How did someone find formulas for the solution functions in part 2a? Or how could we have found them if they weren't given to us?

$$\frac{dy}{dx} = y^2$$

Answer: They used the chain rule in reverse .... the systematic way of doing this is called "separation of variables", section 1.4, which we'll discuss in more detail tomorrow and which many of you discussed in a prerequisite Calculus class. Let's work the "chain rule in reverse" for this example in order to recall ...

• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example, you've seen the *exponential growth/decay differential equation*, modeling situations in which *The rate of change of the quantity* $P(t)$ *is proportional to* $P(t)$:

$$P'(t) = k\,P(t)$$

with solutions

$$P(t) = P_0\, \mathrm{e}^{k\,t}.$$

And you've seen the *constant acceleration* particle motion differential equation

$$y''(t) = a \quad \text{(constant)}$$

with solutions

$$y(t) = y_0 + v_0\, t + \frac{a}{2}t^2.$$

We'll see many more differential equations applications in this class. The general modeling paradigm and feedback loop is discussed in our text in section 1.1:

4 Chapter 1 First-Order Differential Equations

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

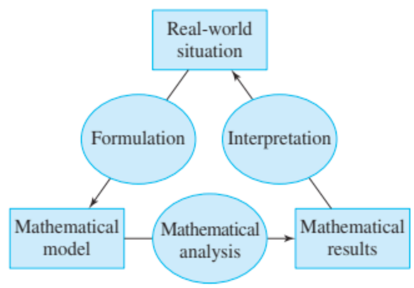


FIGURE 1.1.4. The process of mathematical modeling.

For example, the exponential growth model is effective for continuous compound interest in bank accounts, bacterial growth with no resource constraints, and radioactive decay (negative k). And the constant acceleration model is effective when there are no drag forces on the object and the underlying acceleration is close to a constant. But more sophisticated models are needed if the situation is less simple, e.g. the *logisitic* population model for populations with resource constraints, and particle acceleration models that need to take into account drag forces and non-constant background acceleration force. We will study such modifications in Chapter 2.

As a concrete prototype for how mathematical modeling works, consider:

Exercise 3) *Newton's law of cooling* is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ is assumed to change at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use the assumptions in the model above, to "derive" (i.e. explain) the differential equation for the $T(t)$ of the object being heated or cooled:

$$\frac{dT}{dt} = -k(T - A) .$$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

c) Use the Newton's law of cooling model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70° F. An hour later the body temperature has decreased to 60° . It's been a winter inversion in SLC, with constant ambient temperature 30° . Assuming the Newton's law model, estimate the time of death. Hint: Begin by finding formulas for the functions $T(t)$ that solve this "separable" differential equation.

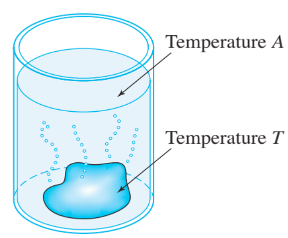


FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Math 2280-1
Tues Jan 8
1.2, 1.4 antidifferentiation differential equations and separable differential equations

Announcements:

Warm-up Exercise:

Section 1.2 is about differential equations equivalent to ones of the form

$$\frac{dy}{dx} = f(x)$$

which we solve by direct antidifferentiation, as you learned in Calculus.

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$

$$y(0) = 0$$

Section 1.4 is about *separable* differential equations which is a generalization that includes those of section 1.2:

Definition: A *separable* first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

Solution (chain-rule justified): One can rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad (\text{as long as } \phi(y) \neq 0) .$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Taking antiderivatives with respect to the variable x we have

$$\int g(y)\frac{dy}{dx}\, dx = \int f(x)\, dx.$$

If $G(y)$ is any antiderivative of $g(y)$ with respect to the variable y then

$$g(y(x))\frac{dy}{dx} = G'(y(x))y'(x)$$

which by the chain rule (read backwards) is precisely

$$\frac{d}{dx}G(y(x)).$$

So we have

$$\int \frac{d}{dx}G(y(x))dx = \int f(x)\, dx ,$$

which we antidifferentiate with respect to x and obtain

$$G(y(x)) = F(x) + C.$$

where $F(x)$ is any particular antiderivative of $f(x)$. This identity

$$G(y) = F(x) + C$$

expresses solutions $y(x)$ *implicitly* as functions of x . (By differentiating this identity implicitly as you did in Calculus, you recover the original differential equation.)

You may be able to use algebra to solve this equation *explicitly* for $y = y(x)$ as we did for $T = T(t)$ in the Newton's Law of cooling problem.

<u>Solution (differential magic for doing the computation quickly)</u>: Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx , and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx\ .$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx\ \text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C\ . \quad \text{Agrees!}$$

This differential magic is related to the "method of substitution" in antidifferentiation, which is essentially the "chain rule in reverse" for integration techniques.

Exercise 2: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 \, .$$

a) Use separation of variables to find solutions to this DE.

b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a?

c) Explain why the IVP

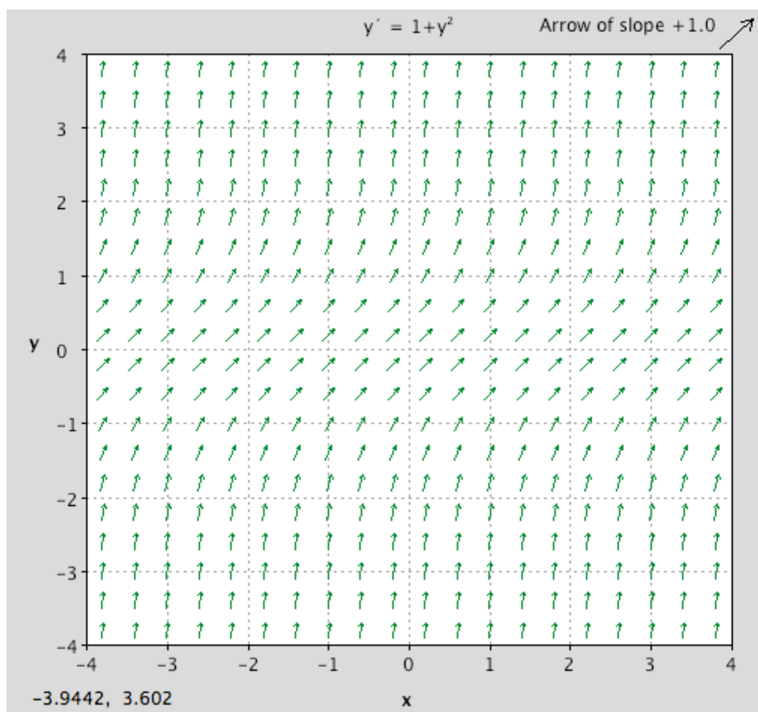
$$\frac{dy}{dx} = 1 + y^2$$
$$y(0) = 0$$

has a solution, but this solution does not exist for all x.

You can download the java applet "dfield" from the Rice University URL

http://math.rice.edu/~dfield/dfpp.html

(You also have to download a toolkit, following the directions there.)



Section 1.2 applications:

An important class of antidifferentiation differential equations applications arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

Exercise 3:

a) If the units for position are meters m and the units for time are seconds s , what are the units for velocity and acceleration? (These are *mks* units.)

b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

Exercise 4: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof. (Use the approximate value for the acceleration due to gravity, $g = 9.8\ \frac{m}{s^2}$.)

a) Neglecting friction, how high will the object get above ground?

b) When does the object land ?

Exercise 5:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a .$$

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{\text{max}} = \frac{1}{2}\frac{v_0^2}{a} .$$

(This formula may help with some homework problems.)

c) Adapt the answer to b to check part of your work in Exercise 4.

Math 2280-002
Wed Jan 9
1.3-1.4   more slope fields; existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Announcements:

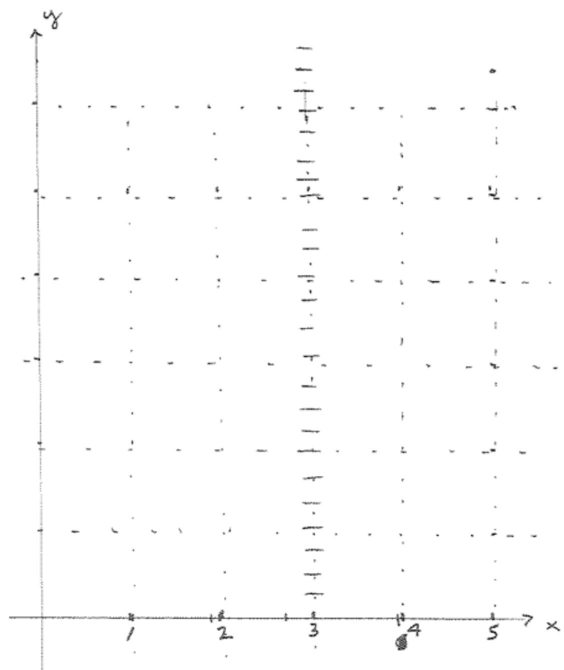
Warm-up Exercise:

Exercise 1: Consider the differential equation $\frac{dy}{dx} = x - 3$, and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \leq x \leq 5, 0 \leq y \leq 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.

b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.

c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).



The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

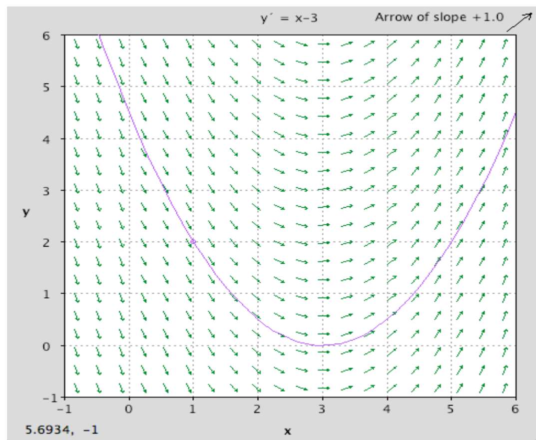
$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0), by picking a step size Δx and then incrementing y by

$$\Delta y = f(x, y)\,\Delta x.$$

In this way one can *approximate* solution functions to initial value problems, and their graphs. The Java applet "dfield" (stands for "direction field", which is a synonym for slope field) uses (a more sophisticated analog of) this method to compute approximate solution graphs.

Here's a picture like the one we sketched by hand on the previous page, created by dfield.



Exercise 2: Consider the IVP

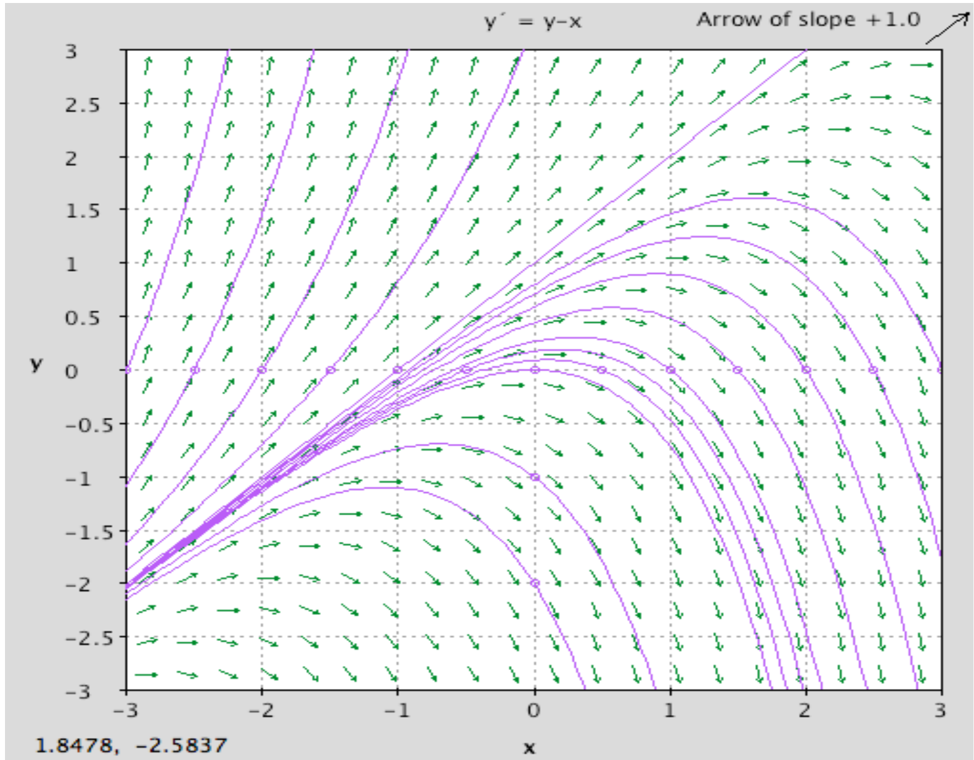
$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $y(x) = x + 1 + C\,e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.

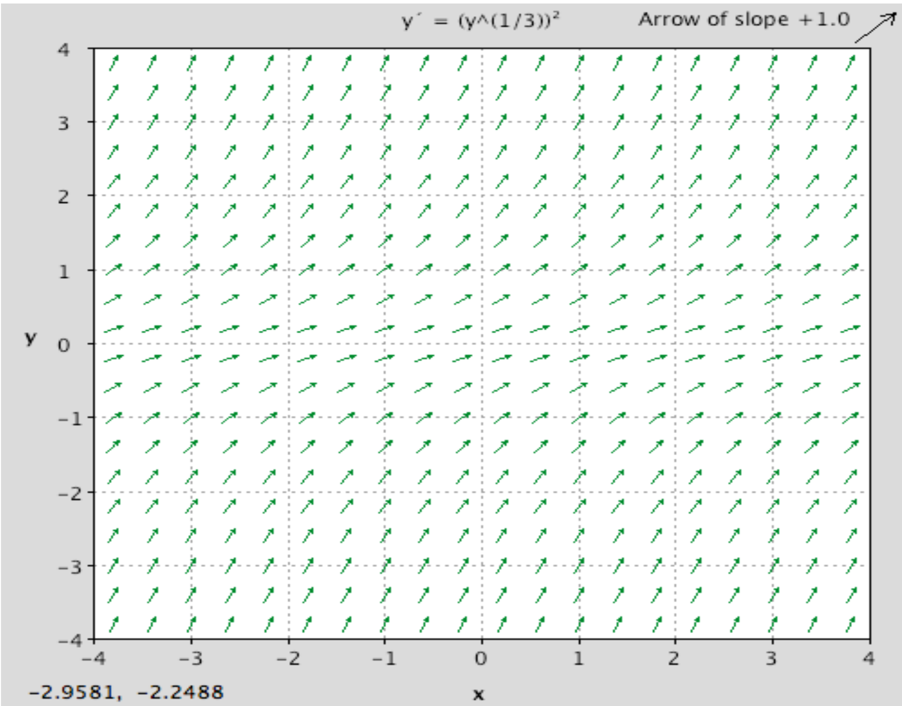

y´ = y-x
Arrow of slope +1.0
y
x
1.8478, -2.5837
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3
−2
−1
0
1
2
3

Exercise 3a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(0) = 0$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

3c) Sketch some of these singular solutions onto the slope field below.

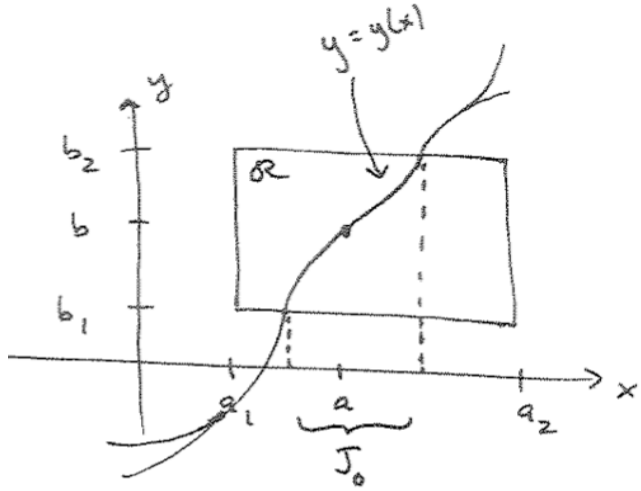


Here's what's going on (stated in 1.3 page 22 of text as *Theorem 1*; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \le x \le a_2, b_1 \le y \le b_2$ in the $x-y$ plane.
- <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 4</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Exercises 1-3 in today's notes.

Math 2280-001
Fri Jan 11
1.3-1.4 more slope fields and existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Announcements:

Warm-up Exercise:

- Review the existence-uniqueness theorem from the end of class Wednesday.

Exercise 1 (A slight variation on Exercise 3 in Wednesday's notes. Also, one of your homework problems is similar.) Consider the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(3) = 8.$$

a) Does the IVP above have a unique solution, according to the existence-uniqueness theorem?

b) Find the IVP solution above, using separation of variables. What is the largest interval on which it is the unique solution?

c) What happens when you solve this DE numerically with dfield?

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice two of these are NOT separable differential equations.) Can Maple (or Wolfram alpha) find formulas for the solution functions?

a)

$$y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$

> $with(DEtools):$

> $dsolve\left(y'(x) = (x+1)\cdot(y(x)-3)^2, y(x)\right);$

$$y(x) = \frac{3\,x^2 + 6\,\_C1 + 6\,x - 2}{x^2 + 2\,\_C1 + 2\,x} \tag{1}$$

>

>

b)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^2 + y(x)^2, y(x)\right);$

$$y(x) = \frac{\left(-\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}\,x^2\right)\_C1 - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}\,x^2\right)\right)x}{\_C1\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}\,x^2\right) + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}\,x^2\right)} \tag{2}$$

>

c)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^4 + y(x)^4, y(x)\right);$

>

For your section 1.2 and 1.4 homework this week I assigned a selection of application problems. Some applications will be familiar to you from previous courses, e.g. exponential growth and Newton's Law of cooling, velocity-acceleration problems. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

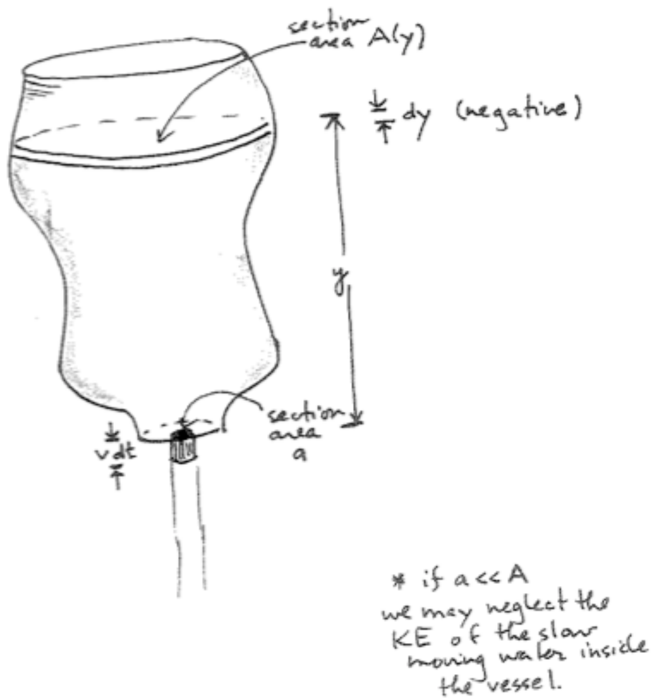
<u>Toricelli's Law</u>, for draining water tanks. Refer to the figure below.
<u>Exercise 3</u>:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity $(a << A)$. Equate the lost potential energy from the top in time *dt* to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed *v* with which the water exits the tank is given by

$$v = \sqrt{2gy}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2g}).$$



Experiment fun!   I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\,y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k than on previous page).

Exercise 4a) Solve the differential equation and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank.

$$\frac{dy}{dt} = -k\,y^{\frac{1}{2}}$$
$$y(0) = 1$$

Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

and the inverse function $t = t(y)$ is given by

$$t = \frac{2}{k}\left(1 - \sqrt{y}\right)$$

(until the tank runs empty).

Exercise 4b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let t_1 be the time it takes the the water to go from height 1 (full) to height 0.5 (half empty). Let t_2 be the time it takes for the water to go from height 1 (full) to height 0.0 (empty).   Show that

$$t_2 = \frac{1}{1 - \sqrt{.5}}\,t_1 \approx 3.41\,t_1.$$

Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

> $Digits := 5:$ *# that should be enough significant digits*

> $\frac{1}{1-\mathrm{sqrt}(.5)}$; *# the factor from previous page*

$$3.4143 \tag{3}$$

> $Thalf := 35;$ *# seconds to half-empty canteen in a previous test.*
> $Tpredict := 3.4143 \cdot Thalf;$ *#prediction*

$$Thalf := 35$$

$$Tpredict := 119.50 \tag{4}$$

>

What are possible defects in our model?

What does the existence-uniqueness theorem say about solutions to IVP's for this DE when the initial height is zero? Does this make sense?

$$\frac{dy}{dt} = -k\,|y|^{\frac{1}{2}}$$

$$y(0) = 0$$