Because, It is always true that if A is a matrix with real entries and

$$A\vec{v} = \lambda\vec{v}$$

then $A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

eigenvalues {{-.1,-.4},{.1,-.1}}

Browse Examples Surprise Me

Input:

eigenvalues $\begin{pmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{pmatrix}$

Open code

Results: Decimal forms Step-by-step solution

$\lambda_1 = -\frac{1}{10} + \frac{i}{5}$

$\lambda_2 = -\frac{1}{10} - \frac{i}{5}$

Corresponding eigenvectors: Step-by-step solution

$v_1 = (2i, 1)$

$v_2 = (-2i, 1)$

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$e^{\lambda t}\vec{v}$

so get complex solns

$$e^{(-.1+.2i)t}\begin{bmatrix} 2i \\ 1 \end{bmatrix}, \quad e^{(-.1-.2i)t}\begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} G' \\ H' \end{bmatrix} = \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix}\begin{bmatrix} G \\ H \end{bmatrix}$$

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\vec{X}(t) = e^{-.1t}\left(\cos(.2t) + i\sin(.2t)\right)\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-.1t}\left(\cos(.2t) + i\sin(.2t)\right)2i \\ e^{-.1t}\left(\cos(.2t) + i\sin(.2t)\right) \end{bmatrix}$$

$$= e^{-.1t}\begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + i\, e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$e^{\lambda t}\vec{v} = \vec{X}(t) = \vec{u}(t) + i\vec{v}(t)$$

as with real eigendata

$$\vec{X}'(t) = A\vec{X}$$
$$(= \lambda e^{\lambda t}\vec{v})$$

$$\vec{X}'(t) = \vec{u}'(t) + i\vec{v}'(t)$$
$$A\vec{X} = A\vec{u} + iA\vec{v}$$

so real & imag parts of $\vec{X}'$ & $A\vec{X}$ must match (using conjugate data gives equivalent results

So circled real vector fcns $\{\vec{u}(t), \vec{v}(t)\}$ are a basis

1d) Solve the initial value problem.

$$\begin{bmatrix} G(t) \\ H(t) \end{bmatrix} = c_1 e^{-.1t} \begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + c_2 e^{-.1t} \begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$@\ t=0 \quad \begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$100 = 2c_2 \qquad c_2 = 50$

$0 = c_1$

$$\begin{bmatrix} G(t) \\ H(t) \end{bmatrix} = 50\, e^{-.1t} \begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix} = \begin{bmatrix} 100\, e^{-.1t}\cos(.2t) \\ 50\sin(.2t) \end{bmatrix}$$

turns ellipse into spiral "sink" because $\left\|\begin{bmatrix} G \\ H \end{bmatrix}\right\| \to 0$. (exponentially)

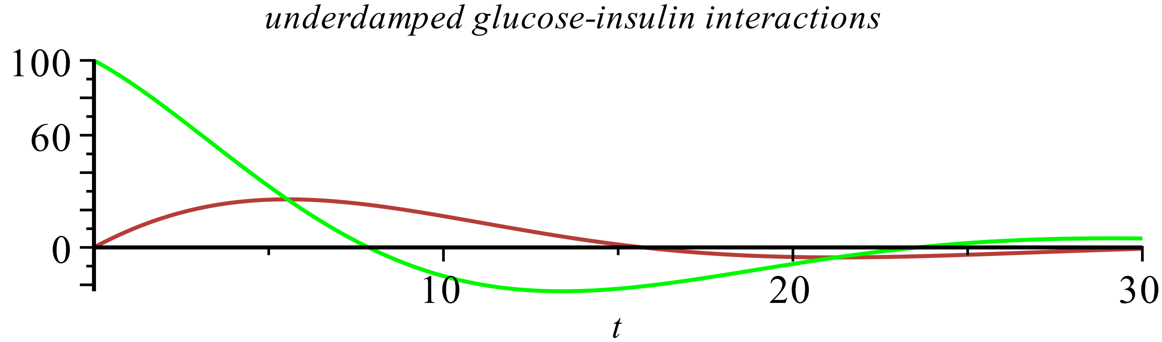
circling an ellipse $\frac{G^2}{4} + H^2 = 1$

Here are some pictures to help understand what the model is predicting ... using the analytic solution formulas we found.

(1) Plots of glucose vs. insulin, at time t hours later:

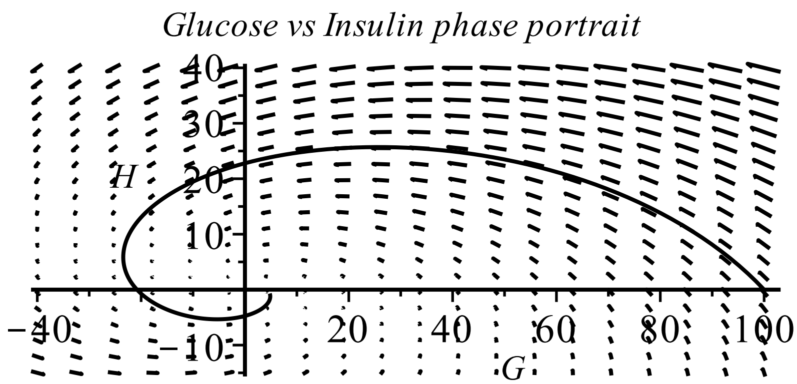
> $with(plots):$

> $G := t \rightarrow 100 \cdot \exp(-.1 \cdot t) \cdot \cos(.2 \cdot t):$
$H := t \rightarrow 50 \cdot \exp(-.1 \cdot t) \cdot \sin(.2 \cdot t):$
$plot1 := plot(G(t), t = 0..30, color = green):$
$plot2 := plot(H(t), t = 0..30, color = brown):$
$display(\{plot1, plot2\}, title = \text{`underdamped glucose-insulin interactions`});$

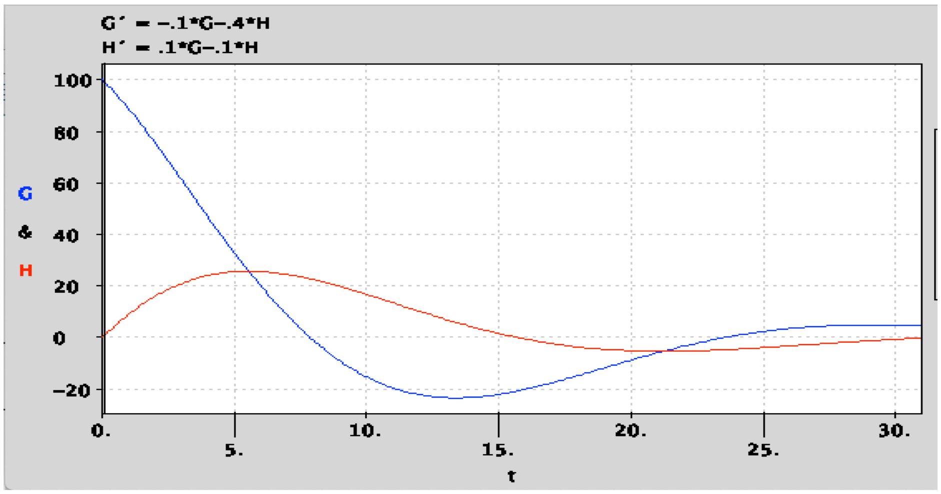


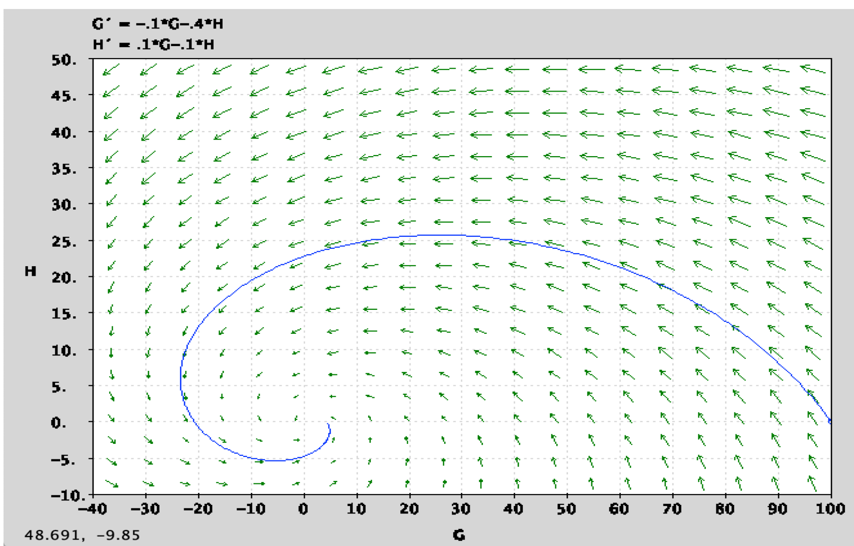
2) A phase portrait of the glucose-insulin system:

> $pict1 := fieldplot([-.1 \cdot G - .4 \cdot H, .1 \cdot G - .1 \cdot H], G = -40..100, H = -15..40):$
$soltn := plot([G(t), H(t), t = 0..30], color = black):$
$display(\{pict1, soltn\}, title = \text{`Glucose vs Insulin phase portrait`});$



Same plots computed numerically using pplane. Numerical solvers for higher order differential equations IVP's convert them to equivalent first order system IVP's, and then use a Runge-Kutta type algorithm to find the solutions to the first order systems, and extract the first component function as the solution to the original differential equation IVP.





now go to Friday's

Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

satisfy $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a complex solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\boldsymbol{x}$$

because $\underline{\boldsymbol{z}}'(t) = \lambda e^{\lambda t}\underline{\boldsymbol{v}}$ and this is equal to $A\,\underline{\boldsymbol{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$.

- But if we write $\underline{\boldsymbol{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\boldsymbol{z}}(t) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)$$

then the equality

$$\underline{\boldsymbol{z}}'(t) = A\,\boldsymbol{z}$$
$$\Rightarrow \underline{\boldsymbol{x}}'(t) + i\,\underline{\boldsymbol{y}}'(t) = A(\underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)) = A\,\underline{\boldsymbol{x}}(t) + i\,A\,\underline{\boldsymbol{y}}(t)\,.$$

Equating the real and imaginary parts on each side yields

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}(t)$$
$$\underline{\boldsymbol{y}}'(t) = A\,\underline{\boldsymbol{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\left(\underline{\alpha} + i\,\underline{\boldsymbol{\beta}}\right) = (a + b\,i)\left(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}}\right)$ then it is straightforward to check that $A\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = (a - b\,i)\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right)$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i\,b)t}\left(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}\right) = \underline{\boldsymbol{x}}(t) - i\,\underline{\boldsymbol{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real soluitions look like:

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

$$\Rightarrow \mathrm{e}^{\lambda t}\underline{\boldsymbol{v}} = \mathrm{e}^{a\,t}(\cos(b\,t) + i\sin(b\,t))\cdot\left(\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}}\,i\right) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t).$$

So the real-valued vector-valued solution functions that we'll use are

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{a\,t}\left(\cos(bt)\underline{\boldsymbol{\alpha}} - \sin(bt)\underline{\boldsymbol{\beta}}\right)$$
$$\underline{\boldsymbol{y}}(t) = \mathrm{e}^{a\,t}\left(\cos(bt)\underline{\boldsymbol{\beta}} + \sin(bt)\underline{\alpha}\right)$$

Complex conjugation for scalars and consequences for matrix/vector products.
March 8, 2019

$\operatorname{Re} z = a, \ \operatorname{Im} z = b$

For $z = a + b\,i$ (with a, b being real numbers) the complex conjugate $\overline{z}$ is defined by $\overline{z} = a - b\,i$.

This algebraic operation is useful in a number of contexts, because it behaves nicely with respect to sums and products of complex numbers:

For $z = a + b\,i$, $w = c + d\,i$ with $a, b, c, d \in \mathbb{R}$, we should check that

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{(a+bi)+(c+di)} = \overline{(a+c)+i(b+d)}$$
$$= (a+c) - i(b+d)$$
$$= (a-ib) + (c-id)$$
$$= \overline{z} + \overline{w}.$$

$$\overline{z\,w} = \overline{z}\,\overline{w}$$

$$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+i(ad+bc)}$$
$$= (ac-bd) - i(ad+bc)$$
$$\overline{z}\,\overline{w} = (a-bi)(c-di) = ac-bd + i(-bc-cd)$$

You compute the complex conjugate of a matrix or vector by computing the conjugate of each entry. Since the individual entries of the sum of two matrices are just the sums of the corresponding entries, it follows that if A, B are two matrices of the same size with complex entries, then

$$\overline{A + B} = \overline{A} + \overline{B}.$$

When you compute the product of two matrices/vectors (or of a scalar times a matrix/vector) entry by entry, you add a bunch of products where each product consists of one term from the first factor, and one term from the second. Using the fact that conjugates of sums are sums of conjugates, and conjugates of products are products of conjugates, it follows that

$$\overline{A\,B} = \overline{A}\,\overline{B}.$$

$$\overline{\begin{bmatrix} 2i+1 & 0 \\ 3-i & 6 \end{bmatrix}\begin{bmatrix} i \\ 2+i \end{bmatrix}} = \overline{\begin{bmatrix} (2i+1)(i) \\ (3-i)(i) + 6(2+i) \end{bmatrix}} \overset{\text{using above}}{=} \begin{bmatrix} (-2i+1)(-i) \\ (3+i)(-i) + 6(2-i) \end{bmatrix}$$

$$\overline{A}\,\overline{B} = \begin{bmatrix} -2i+1 & 0 \\ 3+i & 6 \end{bmatrix}\begin{bmatrix} -i \\ 2-i \end{bmatrix}$$

Math 2280-002
Fri Mar 8
5.3, phase portraits for complex eigendata; Introduction to 6.1: phase diagrams for two linear systems of *nonlinear* first order differential equations

Announcements: $\mathbb{C}$ today. (First finish glucose insulin model, then (skip Wed notes, do Friday's, complete Wed after break)

- also, handout on complex conjugation algebra for scalars & for matrix/vector sums & products

Warm-up Exercise: not today. ☺

complex eigenvalues: Consider the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let $A_{2\times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i$.

Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\begin{aligned}\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= \left[e^{p\,t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p\,t}\sin(q\,t)\underline{\boldsymbol{b}}\right]\\ &\quad + i\left[e^{p\,t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p\,t}\cos(q\,t)\underline{\boldsymbol{b}}\right].\end{aligned}$$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above:

$$\begin{aligned}\underline{\boldsymbol{x}}(t) &= c_1 e^{p\,t}[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}]\\ &\quad + c_2 e^{p\,t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}].\end{aligned}$$

$$= e^{pt}\left[\vec{a} \;\vdots\; \vec{b}\right]\underbrace{\begin{bmatrix}\cos qt & \sin qt\\ -\sin qt & \cos qt\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}}$$

$$\left[\cos qt\,\vec{a} - \sin qt\,\vec{b} \;\vdots\; \sin qt\,\vec{a} + \cos qt\,\vec{b}\right]$$

We can rewrite $\underline{\boldsymbol{x}}(t)$ as

$$\underline{\boldsymbol{x}}(t) = e^{p\,t}\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}\underbrace{\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}}.$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

$$\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

$$[Rot_\theta] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

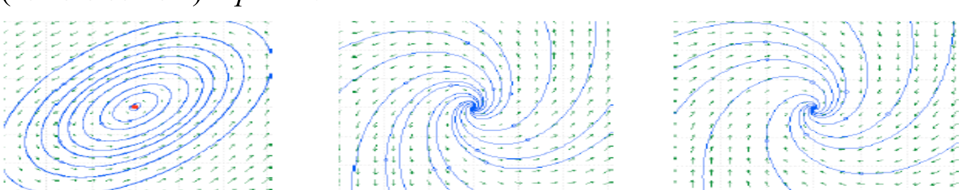
(2270)

- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

$$\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p\,t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

Thus $\underline{\boldsymbol{x}}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center $\mathrm{Re}(\lambda)=0$ spiral source $\mathrm{Re}(\lambda)>0$ spiral sink $\mathrm{Re}(\lambda)<0$

Exercise 1) Do the eigenvalue analysis, ~~find the general solution~~, and use tangent vectors just along the two axes to sketch typical solution curve trajectories, for this system from your posponed homework problem.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$\begin{bmatrix} x \\ v \end{bmatrix}$	$\begin{bmatrix} x' \\ v' \end{bmatrix} = A \begin{bmatrix} x \\ v \end{bmatrix}$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -5 \end{bmatrix}$
$\begin{bmatrix} c \\ 0 \end{bmatrix}$	$c\begin{bmatrix} 0 \\ -5 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$d\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$d\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

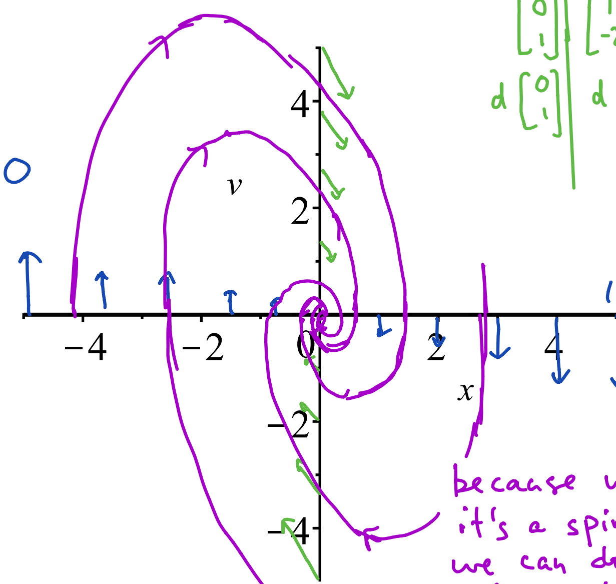
$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2)+5$$
$$= \lambda^2+2\lambda+5$$
$$= (\lambda+1)^2+4 = 0$$
$$(\lambda+1)^2 = -4$$
$$\lambda+1 = \pm 2i$$
$$\lambda = -1 \pm 2i$$

$e^{\lambda t}\vec{v}$

$e^{(-1+2i)t}\vec{v}$

$e^{(-1-2i)t}\bar{\vec{v}}$

e^{-t} factor says spiral sink.



$(5,0)$ $\quad 5\begin{bmatrix} 0 \\ -5 \end{bmatrix}$

because we know it's a spiral sink we can do a good job of interpolating the solution curves just with the partial tangent info along the coord axes.