For example, consider this second order underdamped IVP for $x(t)$:

$$\begin{cases} x'' + 2x' + 5x = 0 \\ x(0) = 4 \\ x'(0) = -4 \ . \end{cases}$$

Exercise 1)

1a) Suppose that $x(t)$ solves the IVP above. Show that $[x(t), x'(t)]^T$ solves the first order system initial value problem below as in discussions yesterday.

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}. \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix}$$

$$x'' = -5x - 2x'$$

$$\Longrightarrow \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}' = \begin{bmatrix} x' \\ -5x - 2x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\& \quad \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

1b) Solve the second order IVP in order to deduce the solution to the first order IVP in 5a.

$x(t) = 4e^{-t}\cos 2t$ solves 2nd order DE.

sol is
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 4e^{-t}\cos 2t \\ 4(-e^{-t})\cos 2t - 8e^{-t}\sin 2t \end{bmatrix}$$
$$= 4e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{bmatrix}$$

1c) Start trying to solve the first order system directly, using eigenvalues and eigenvectors. Get at least as far as the roots of the characteristic polynomial. You'll finish this problem in your homework in order to practice working with complex eigenvectors, Euler's formula, and with the fact that higher order DE's correspond to certain first order systems of DE's. (It's postponed until after break.) You'll eventually recover the solution 1b as the first entry in the vector function solution to 1a.

In HW: Solve
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

using eigendata method.

then $x_1(t)$ will solve 2nd order DE

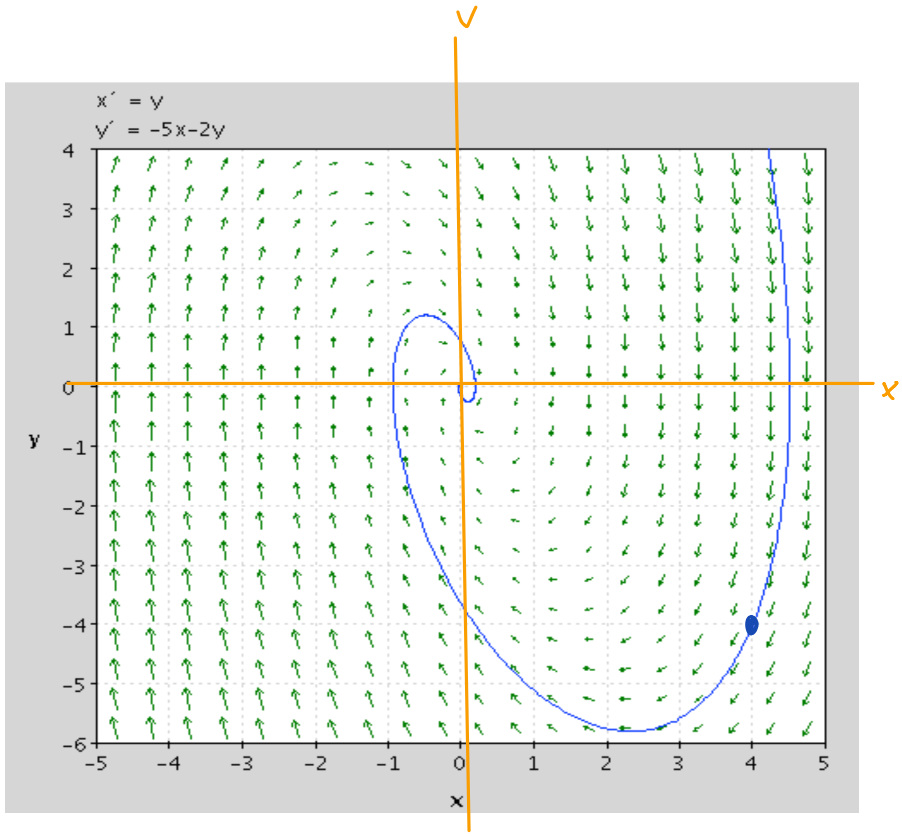
find eigenvals of A

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 5 = \lambda^2 + 2\lambda + 5$$

($p(r) = r^2 + 2r + 5$ for DE!)

roots $\lambda = -1 \pm 2i$

1d) This phase portrait and solution curve is for $[x(t), x'(t)]^T = [x(t), v(t)]^T$ from the original second order DE, corresponding to the first order system we were discussing. Interpret the solution curve in terms of the underdamped motion.



Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} -.1G - .4H \\ .1G - .1H \end{bmatrix}$$

Exercise 1a) Understand why the signs of the matrix entries are reasonable.

$a_{11} < 0$: if $H = 0$ & if $G > 0$, $G'(t) < 0$ because background insulin metabolizes excess glucose

$a_{12} = -.4 < 0$ if $H > 0$ then glucose metabolized more quickly

$a_{21} = .1 > 0$ if excess glucose, pancreas notices, secretes more insulin

$a_{22} = -.1 < 0$, background glucose is causing insulin to get used.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

1b) The first step is to get the eigendata of the matrix. Do this, and compare with the Wolfram alpha check on the next page.

$$\begin{vmatrix} -.1-\lambda & -.4 \\ .1 & -.1-\lambda \end{vmatrix} = (\lambda+.1)^2 + .04 = 0$$

$$(\lambda+.1)^2 = -.04 \qquad \lambda + .1 = \pm .2i$$

$$\lambda = -.1 \pm .2i$$

$\lambda = -.1 + .2i$

$$\left[\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \end{array}\right]$$

$$\begin{array}{l} 10R_2 \\ 5R_1 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ -i & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ iR_1 + R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$i(-2i) - 2 = 2 - 2 = 0!$

So, $E_{\lambda = -.1+.2i} = span\left\{\begin{bmatrix} 2i \\ 1 \end{bmatrix}\right\}$

$\lambda = -.1 - .2i$

$$\left[\begin{array}{cc|c} .2i & -.4 & 0 \\ .1 & .2i & 0 \end{array}\right]$$

$$\begin{array}{l} 10R_2 \to R_1 \\ 5R_1 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & 2i & 0 \\ i & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ -iR_1 + R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $E_{\lambda = -.1-.2i} = span\left\{\begin{bmatrix} -2i \\ 1 \end{bmatrix}\right\}$

NOTE: For the conjugate eigenvalue $\lambda = -.1 - .2i$ we can use the conjugate eigenvector $\begin{bmatrix} -2i \\ 1 \end{bmatrix}$.

Because, It is always true that if A is a matrix with real entries and

$$A\vec{v} = \lambda\vec{v}$$

then $A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

eigenvalues {{-.1,-.4},{.1,-.1}}

Browse Examples Surprise Me

Input:

eigenvalues $\begin{pmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{pmatrix}$

Open code

Results: Decimal forms Step-by-step solution

$\lambda_1 = -\frac{1}{10} + \frac{i}{5}$

$\lambda_2 = -\frac{1}{10} - \frac{i}{5}$

Corresponding eigenvectors: Step-by-step solution

$v_1 = (2\,i, 1)$

$v_2 = (-2\,i, 1)$

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$e^{\lambda t}\vec{v}$

So get complex solns

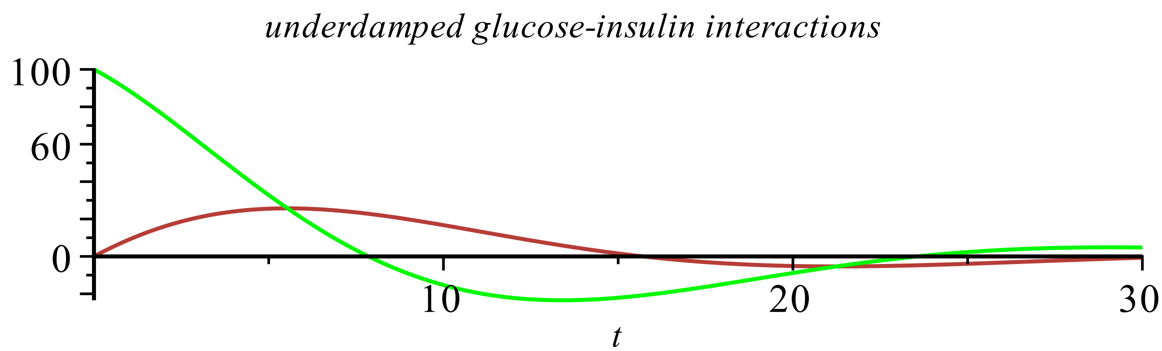
$$e^{(-.1+.2i)t}\begin{bmatrix} 2i \\ 1 \end{bmatrix}, \quad e^{(-.1-.2i)t}\begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

1d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... using the analytic solution formulas we found.

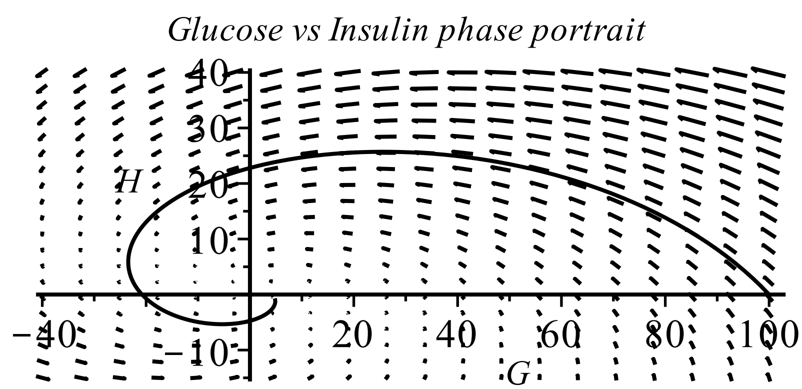
(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t→100·exp(−.1·t)·cos(.2·t) :
  H := t→50·exp(−.1·t)·sin(.2·t) :
  plot1 := plot(G(t), t = 0 ..30, color = green) :
  plot2 := plot(H(t), t = 0 ..30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

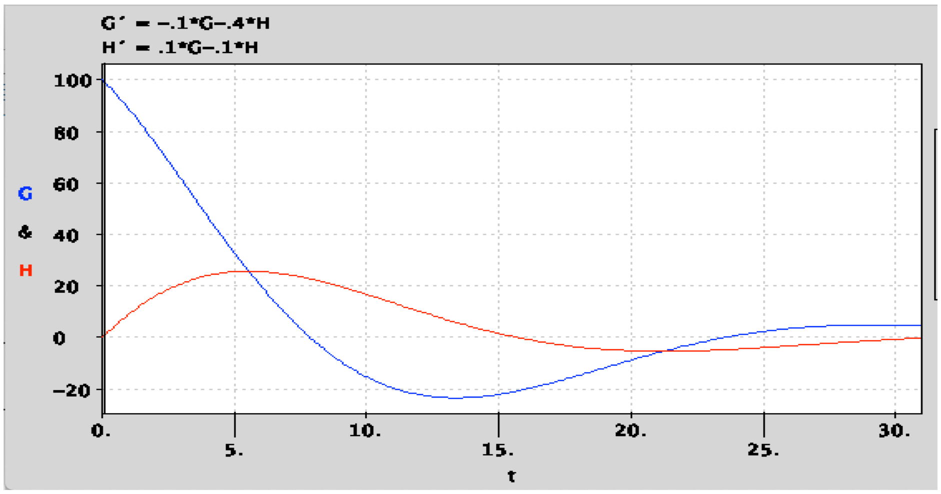


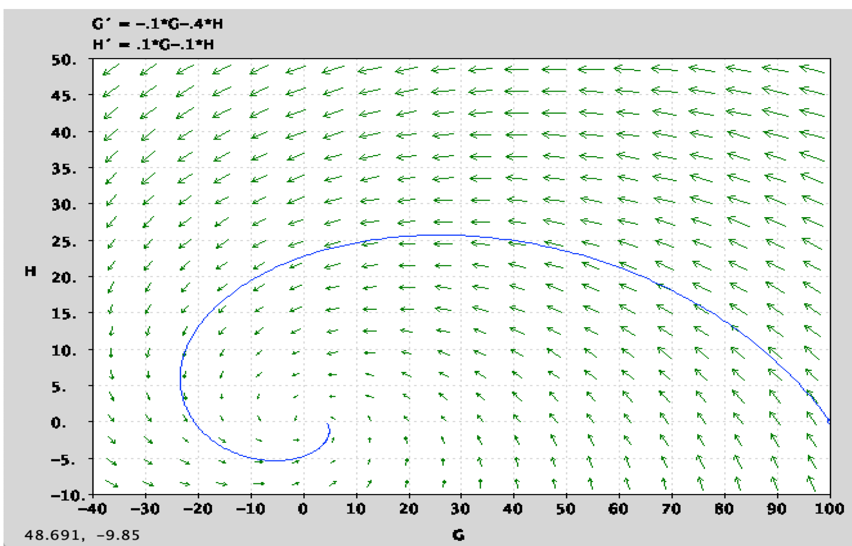
2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([−.1·G − .4·H, .1·G − .1·H], G =−40 ..100, H =−15 ..40) :
  soltn := plot([G(t), H(t), t = 0 ..30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```



Same plots computed numerically using pplane. Numerical solvers for higher order differential equations IVP's convert them to equivalent first order system IVP's, and then use a Runge-Kutta type algorithm to find the solutions to the first order systems, and extract the first component function as the solution to the original differential equation IVP.





Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \;\in \mathbb{C}^n$$

satisfy $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a complex solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\boldsymbol{x}$$

because $\underline{\boldsymbol{z}}'(t) = \lambda e^{\lambda t}\,\underline{\boldsymbol{v}}$ and this is equal to $A\,\underline{\boldsymbol{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$.

- But if we write $\underline{\boldsymbol{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\boldsymbol{z}}(t) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)$$

then the equality

$$\underline{\boldsymbol{z}}'(t) = A\,\boldsymbol{z}$$
$$\Rightarrow \underline{\boldsymbol{x}}'(t) + i\,\underline{\boldsymbol{y}}'(t) = A(\underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)) = A\,\underline{\boldsymbol{x}}(t) + i\,A\,\underline{\boldsymbol{y}}(t)\ .$$

Equating the real and imaginary parts on each side yields

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}(t)$$
$$\underline{\boldsymbol{y}}'(t) = A\,\underline{\boldsymbol{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\,(\underline{\alpha} + i\,\underline{\boldsymbol{\beta}}) = (a + b\,i)(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}})$ then it is straightforward to check that $A\,(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = (a - b\,i)(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a\,-\,i\,b)t}(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = \underline{\boldsymbol{x}}(t) - i\,\underline{\boldsymbol{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real soluitions look like:

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \;\in \mathbb{C}^n$$

$$\Rightarrow \mathrm{e}^{\lambda t}\underline{\boldsymbol{v}} = \mathrm{e}^{a\,t}(\cos(b\,t) + i\sin(b\,t))\cdot(\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}}\,i) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t).$$

So the real-valued vector-valued solution functions that we'll use are

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{a\,t}\big(\cos(bt)\underline{\boldsymbol{\alpha}} - \sin(bt)\underline{\boldsymbol{\beta}}\big)$$
$$\underline{\boldsymbol{y}}(t) = \mathrm{e}^{a\,t}\big(\cos(bt)\underline{\boldsymbol{\beta}} + \sin(bt)\underline{\alpha}\big)$$