Theorem 4) The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and the tank example): For the homogeneous system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

with $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$, $A_{n \times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n\}$ of $\mathbb{R}^n$ made out of eigenvectors of A, i.e. $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ for each $j = 1, 2, \dots, n$), then the functions

$$e^{\lambda_j t}\,\underline{\boldsymbol{v}}_j\,, \quad j = 1, 2, \dots n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\underline{\boldsymbol{x}}_H(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \dots + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n\,.$$

proof: check the Wronskian matrix at $t = 0$, it's the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

if $\vec{X}_j(t) = e^{\lambda_j t}\,\vec{v}_j$ where $A\vec{v}_j = \lambda_j \vec{v}_j$

then $\vec{X}_j'(t) = \lambda_j e^{\lambda_j t}\,\vec{v}_j \quad \left(+ e^{\lambda_j t}\,\vec{0} \right)$ ← LHS

$A\vec{X}_j(t) = A\underbrace{e^{\lambda_j t}}_{\text{scalar}}\vec{v}_j = e^{\lambda_j t} A\vec{v}_j = e^{\lambda_j t}\lambda_j\vec{v}_j$ ← RHS

LHS = RHS

solve IVP's at $t=0$, get

at $t = 0$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \overset{?}{=} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

uniquely solvable.

Discussed on Tuesday 3/5

There is an alternate direct proof of Theorem 4 which is based on diagonalization from Math 2270. You are asked to use the idea of this alternate proof to solve a nonhomogeneous linear system of DE's this week - in homework problem w8.4e.

proof 2:

Definition: Let $A_{n\times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the invertible matrix with those eigenvectors as columns:

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 | \lambda_2\underline{\boldsymbol{v}}_2 | \cdots | \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1} \qquad \text{mult on right by } P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}. \qquad \text{mult on left by } P^{-1}$$

w8.4

$\vec{f}(t)$ = const vector

Now let's feed this into our system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

$$\vec{x}'(t) = A\vec{x} + \vec{f}(t)$$

let's change functions in our DE system:

$$\underline{\boldsymbol{x}}(t) = P\,\underline{\boldsymbol{u}}(t), \qquad (\ \underline{\boldsymbol{u}}(t) = P^{-1}\underline{\boldsymbol{x}}(t)\ )$$

$$\vec{x}(t) = P\vec{u}(t)$$

and work out what happens (and see what would happen if the system of DE's was non-homogeneous, as in your homework).

$$(P\vec{u}(t))' = AP\vec{u}(t)$$

$$P\vec{u}'(t) = AP\vec{u}(t)$$

P^{-1} on left $\quad \vec{u}'(t) = \underbrace{P^{-1}AP}_{D}\vec{u}$

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$(P\vec{u})' = AP\vec{u} + \vec{f}(t)$$

$$P\vec{u}' = AP\vec{u} + \vec{f}(t)$$

$$\vec{u}' = \underbrace{P^{-1}AP}_{D}\vec{u} + \underbrace{P^{-1}\vec{f}}_{\vec{g}(t)}$$

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 + g_1(t) \\ \lambda_2 u_2 + g_2(t) \\ \vdots \\ \lambda_n u_n + g_n(t) \end{bmatrix}$$

$$u_1' = \lambda_1 u_1$$
$$u_2' = \lambda_2 u_2$$
$$\vdots$$
$$u_n' = \lambda_n u_n$$

constants in HW (constant coeff 1st order linear DE's are our favorites — Chapter 1 & Chapter 3)

So Chptr 1

$$u_1(t) = c_1 e^{\lambda_1 t}$$
$$u_2(t) = c_2 e^{\lambda_2 t}$$
$$\vdots$$
$$u_n(t) = c_n e^{\lambda_n t}$$

So

$$\vec{x}(t) = P\,\vec{u}(t)$$

$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \cdots + c_n e^{\lambda_n t} \vec{v}_n$$

(same!)

1d) Find the general solution to the first order homogeneous DE in this problem, using the eigendata method we talked about last week. Note the following correspondences, which verify the discussion in the previous parts:

(i) The first component $x_1(t)$ *is* the general solution of the original second order homogeneous DE that we started with.

(ii) The eigenvalue "characteristic equation" for the first order system *is* the same as the "characteristic equation" for the second order DE.

(iii) The "Wronskian matrix" for the first order system *is* a "Wronskian matrix" for the second order DE.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$

$$= \lambda(7+\lambda) + 6$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) \longleftarrow \text{"same" charac. poly as for DE}$$

evals $\lambda = -1, -6$.

$$E_{\lambda=-1} = Nul \begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad 1 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \vec{0}$$

$$E_{\lambda=-6} = Nul \begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ -6 \end{bmatrix} \right\} \qquad 1 \begin{bmatrix} 6 \\ -6 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$$

$$\vec{x}_H(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-6t} \\ -c_1 e^{-t} - 6c_2 e^{-6t} \end{bmatrix}$$

for IVP in (b),
solt'n was, from above

$$\begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix} = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$= 2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} - 1 \begin{bmatrix} e^{-6t} \\ -6e^{-6t} \end{bmatrix}$$

same as W for 2nd order DE

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \longrightarrow \quad = \begin{bmatrix} e^{-t} & e^{-6t} \\ -e^{-t} & -6e^{-6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

discussed on Tuesday 3/5

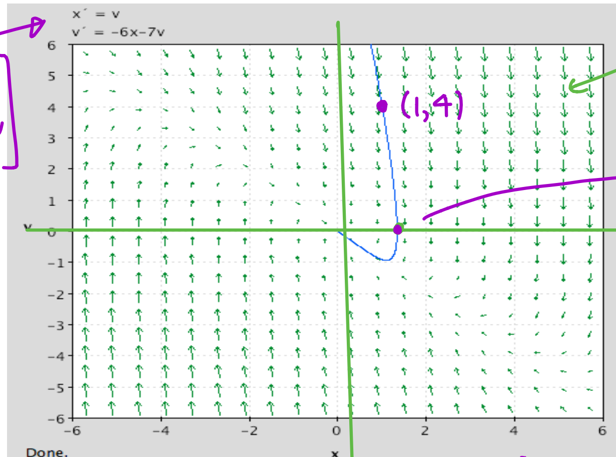
Pictures of the phase portrait and solution curve for the system in 1b, which is tracking position and velocity of the solution to 1a. It's possible to understand the geometry of solution curves in terms of the the eigendata of the coefficient matrix, as we'll demonstrate.

From pplane, for the system:

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$x(0) = 1$

$x'(0) = 4$



"same"

From Wolfram alpha, for the ~~underdamped~~ overdamped second order DE in 3a.

Input:

$\{x''(t) + 7x'(t) + 6x(t) = 0,\ x(0) = 1,\ x'(0) = 4\}$

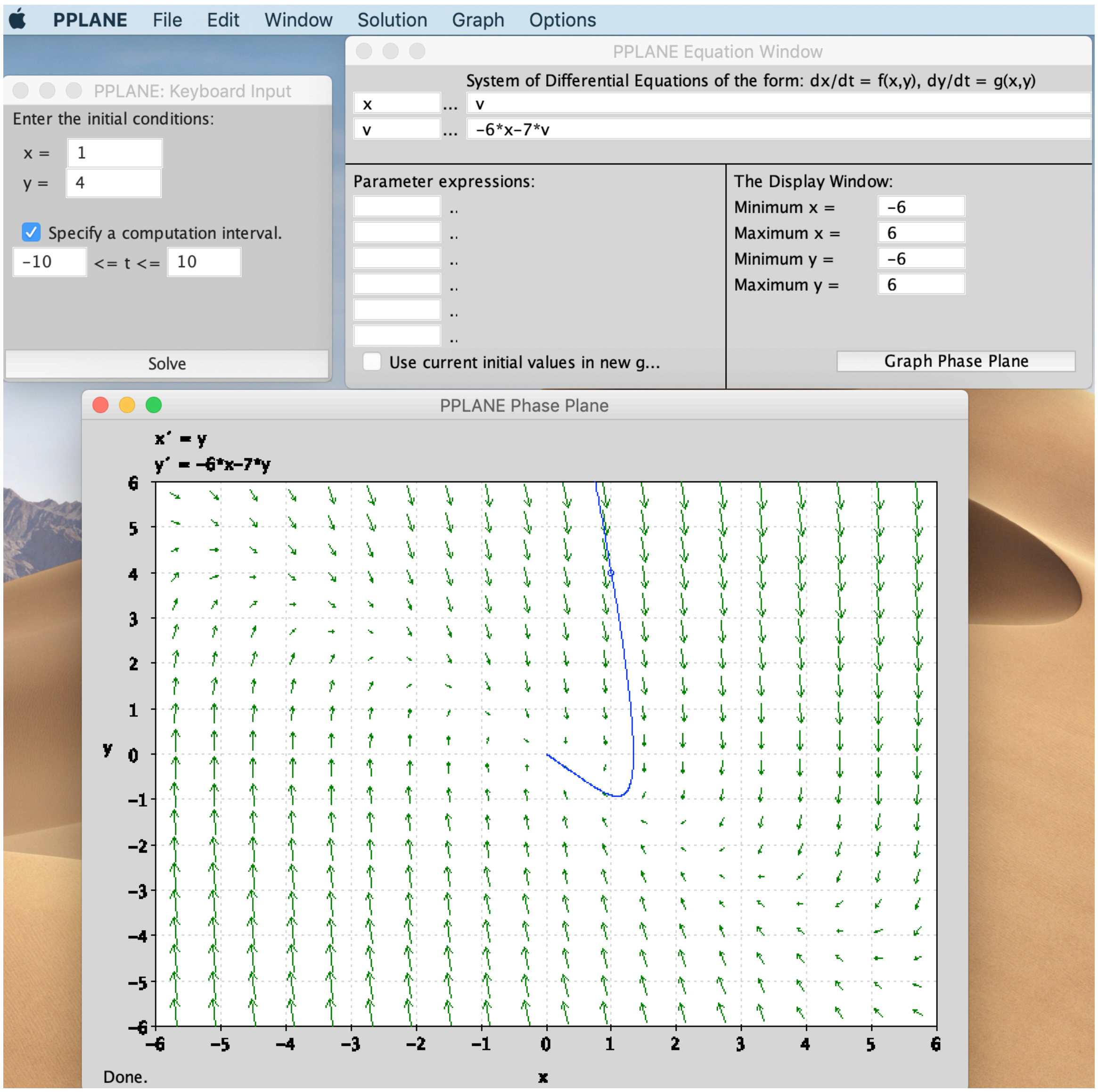
Differential equation solution:

$x(t) = e^{-6t}(2e^{5t} - 1) = 2e^{-t} - e^{-6t}$

Plots of the solution:

We'll demo "pplane". If you don't want to download it to your laptop (from the same URL that had "dfield"), *you can just type "pplane" into a terminal window on the Math Department computer lab computers, and a cloned version of pplane will open. Ask lab assistants (or me) for help, if necessary.*

Here's one screen shot for the system we've focused on, with several of the windows open. There are other interesting visualization options available in pplane that should be helpful for understanding what's going on, as I'll demonstrate in class.



General case of converting a single n^{th} order differential equation for a function $x(t)$ into a first order system of differential equations:

Write the n^{th} order DE in the form:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots x^{(n-1)}(t)\right)$$

and the initial conditions as

$$\begin{aligned} x(t_0) &= b_0 \\ x'(t_0) &= b_1 \\ x''(t_0) &= b_2 \\ &\vdots \\ x^{(n-1)}(t_0) &= b_{n-1}. \end{aligned}$$

Exercise 2a) Show that if $x(t)$ solves the IVP above, then the vector function $\left[x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)\right]^T$ solves the first order system of DE's IVP

$$\begin{aligned} x_1{}' &= x_2 \\ x_2{}' &= x_3 \\ &\vdots \\ x_{n-1}{}' &= x_n \\ x_n{}' &= f\left(t, x_1, x_2, \ldots x_{n-1}\right) \\ x_1(t_0) &= b_0 \\ x_2(t_0) &= b_1 \\ x_3(t_0) &= b_2 \\ &\vdots \\ x_n(t_0) &= b_{n-1} \end{aligned}$$

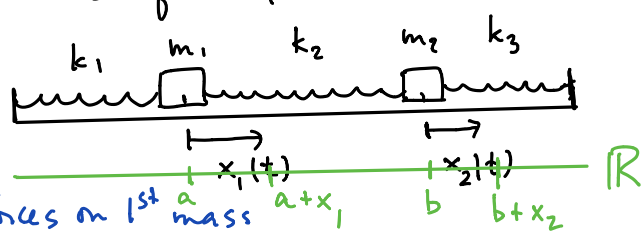
2b) (reversibility): Show that if $\left[x_1(t), x_2(t), \ldots x_n(t)\right]$ is a solution to the IVP in 2a, then the first function $x_1(t)$ solves the original IVP for the n^{th} order differential equtation.

Tues Mar 5
5.2 Linear systems of DE's with complex eigendata

Announcements: w 8.4e, pplane } Monday notes

then maybe begin Tues. notes

Warm-up Exercise: This is HW 4.1.30. We'll find the DE for $x_1(t)$, which is half of the problem.



[k_1, k_2, k_3 are linearization constants (Hookes)

$x_1(t), x_2(t)$ are the displacements of m_1, m_2 from their "equilibrium" positions, i.e. when the system is at rest.

$m_1 x_1'' =$ net forces on 1st mass

$m_2 x_2'' =$ net forces on 2nd mass

e.g.

$$m_1 x_1'' = 0 + F_{spring\,1} + F_{spring}$$

0: net forces at equil.

$F_{spring\,1}$: extra force as x_1, x_2 change

F_{spring}: extra forces

$$m_1 x_1'' = -k_1 x_1 + k_2(\text{spring 2 extra stretch})$$
$$= -k_1 x_1 + k_2(x_2 - x_1)$$

note if $x_2 = x_1$ middle has not extra stretching makes sense

⋮

use $\mathbb{R}$ line to measure the new dist btwn masses

$$= (b + x_2) - (a + x_1)$$
$$= \underbrace{(b-a)}_{\text{stretch at equilibrium}} + \underbrace{(x_2 - x_1)}_{\text{change in stretch of 2nd spring}}$$