Math 2280-002
Week 9, March 4-8 5.1-5.3, introduction to Chapter 6
Mon Mar 4
5.1-5.2 Systems of differential equations - summary so far; converting differential equations (or systems) into equivalent first order systems of DEs; example with pplane visualization and demo.

Announcements: w 8.3 is postponed until next assignment
(What to do with complex eigendata when trying
to solve $\vec{x}'(t) = A\vec{x}$)

Warm-up Exercise: look over first few pages of "review". Any questions on that material?

Summary of Chapters 4-5 so far (for reference):

Theorem 1 For the IVP

$$\underline{\mathbf{x}}'(t) = \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t))$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

If $\underline{\mathbf{F}}(t, \underline{\mathbf{x}})$ is continuous in the t-variable and differentiable in its $\underline{\mathbf{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\mathbf{x}}'(t) + P(t)\underline{\mathbf{x}}(t) = \underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

If the matrix $P(t)$ and the vector function $\underline{\mathbf{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval.

Theorem 3) Vector space theory for first order systems of linear DEs (We noticed the familiar themes... we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\mathbf{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\mathbf{x}}(t)) := \underline{\mathbf{x}}'(t) + P(t)\underline{\mathbf{x}}(t)$$

is linear, i.e.

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{z}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{z}}(t))$$
$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\mathbf{x}}'(t) + P(t)\underline{\mathbf{x}}(t) = \underline{\mathbf{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_p(t) + \underline{\mathbf{x}}_H(t)$$

where $\underline{\mathbf{x}}_p(t)$ is any single particular solution and $\underline{\mathbf{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\mathbf{x}}'(t) + P(t)\underline{\mathbf{x}}(t) = \underline{\mathbf{0}}.$$

3.3) (Generalizes what we talked about on Friday last week.) For $P(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$ defined by

$$[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \ldots, \underline{\boldsymbol{X}}_n)](t_0) := [\underline{\boldsymbol{X}}_1(t_0) | \underline{\boldsymbol{X}}_2(t_0) | \cdots | \underline{\boldsymbol{X}}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible Wronskian matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}'(t) + P(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + \ldots + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients comprise the solution vector to the Wronskian matrix equation

to solve IVP at t_0, solve this

$$\begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, the vector functions $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and the number n is the dimension of the solution space.

New today: Along with the sum rule and constant scalar multiple rule for vector (or matrix) valued functions, we will be using the Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\mathbf{x}}(t)$ (scalar function times vector function),
$f(t)A(t)$ (scalar function times matrix function),
$A(t)\underline{\mathbf{x}}(t)$ (matrix function times vector function),
$\underline{\mathbf{x}}(t) \cdot \underline{\mathbf{y}}(t)$ (vector function dot product with vector function),
$\underline{\mathbf{x}}(t) \times \underline{\mathbf{y}}(t)$ (cross product of two vector functions),
$A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t), B(t)$ be differentiable scalar, matrix or vector-valued functions of t, and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t)*B(t)) = A'(t)*B(t) + A(t)*B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. There is one explanation that proves all of those product rules at once:

example

$$\frac{d}{dt}\begin{bmatrix} t & \cos 3t \\ 17 & e^{10t}\end{bmatrix}\begin{bmatrix} 12t^2+7 \\ 6\sin 4t\end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -3\sin 3t \\ 0 & 10e^{10t}\end{bmatrix}\begin{bmatrix} 12t^2+7 \\ 6\sin 4t\end{bmatrix}}_{A'\ B} + \underbrace{\begin{bmatrix} t & \cos 3t \\ 17 & e^{10t}\end{bmatrix}\begin{bmatrix} 24t \\ 24\cos 4t\end{bmatrix}}_{A\ B'}$$

$$\frac{d}{dt}(A*B)(t)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[A(t+\Delta t)*B(t+\Delta t) - A(t)*B(t)\right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\underbrace{A(t+\Delta t)*B(t+\Delta t) - A(t)*B(t+\Delta t)}_{(A(t+\Delta t)-A(t))*B(t+\Delta t)} + \underbrace{A(t)*B(t+\Delta t) - A(t)*B(t)}_{A(t)*[B(t+\Delta t)-B(t)]}\right]$$

$$= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\left[A(t+\Delta t)-A(t)\right]*B(t+\Delta t) + A(t)*\left(\frac{1}{\Delta t}(B(t+\Delta t)-B(t))\right)\right]$$

limit of sums are sums of limits & limits of products are products of limits

$$A'(t)*B(t) + A(t)*B'(t)$$

Theorem 4) The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and the tank example): For the homogeneous system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

with $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$, $A_{n \times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ of $\mathbb{R}^n$ made out of eigenvectors of A, i.e. $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ for each $j = 1, 2, \ldots, n$), then the functions

$$e^{\lambda_j t}\,\underline{\boldsymbol{v}}_j\,, \quad j = 1, 2, \ldots n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\underline{\boldsymbol{x}}_H(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \ldots + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n\,.$$

proof: check the Wronskian matrix at $t = 0$, it's the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

if $\vec{X}_j(t) = e^{\lambda_j t}\,\vec{v}_j$ where $A\vec{v}_j = \lambda_j \vec{v}_j$

then $\vec{X}_j'(t) = \lambda_j e^{\lambda_j t}\,\vec{v}_j \quad \left(+\ e^{\lambda_j t}\,\vec{0}\right)$ ← LHS

$A\vec{X}_j(t) = A\, e^{\lambda_j t}\,\vec{v}_j = e^{\lambda_j t} A\vec{v}_j = e^{\lambda_j t}\lambda_j \vec{v}_j$ ← RHS

LHS = RHS

↑ scalar

solve IVP's at $t=0$, get

at $t=0$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

uniquely solvable.

There is an alternate direct proof of Theorem 4 which is based on diagonalization from Math 2270. You are asked to use the idea of this alternate proof to solve a nonhomogeneous linear system of DE's this week - in homework problem w8.4e.

proof 2:

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the invertible matrix with those eigenvectors as columns:

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1} A\,P = \mathrm{D}.$$

Now let's feed this into our system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

let's change functions in our DE system:

$$\underline{\boldsymbol{x}}(t) = P\,\underline{\boldsymbol{u}}(t), \qquad \left(\ \underline{\boldsymbol{u}}(t) = P^{-1}\underline{\boldsymbol{x}}(t)\ \right)$$

and work out what happens (and see what would happen if the system of DE's was non-homogeneous, as in your homework).

New today: It is always the case that an initial value problem for one or more differential equations of arbitrary order is equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example. (See examples and homework problems in section 4.1) This gives us a new perspective on the way we solved differential equations from Chapter 3.

For example, consider this overdamped problem from Chapter 3:

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

Exercise 1a) Do enough checking to verify that $x(t) = 2e^{-t} - e^{-6t}$ is the solution to this IVP.

check

$p(r) = r^2 + 7r + 6$
$= (r+6)(r+1)$

$x_H(t) = c_1 e^{-t} + c_2 e^{-6t}$ ✓

$x(0) = 2 - 1 = 1$
$x'(0) = -2 + 6 = 4$ ✓

1b) Show without using the formula for its solution, that if $x(t)$ solves the IVP above, then the vector function $[x(t), x'(t)]^T$ solves the first order system of DE's IVP

IVP 2
$$\left\{ \begin{array}{l} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \end{array} \right.$$

Then use the solution from 1a to write down the solution to the IVP in 1b.

check

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix}$$

but $x''(t) = -6x - 7x'$

$$= \begin{bmatrix} x' \\ -6x - 7x' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So soln to IVP2 is

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

1c) Show that if $[x_1(t), x_2(t)]^T$ solves the IVP in 1b then without using the formula for the solution, show that the first entry $x_1(t)$ solves the original second order DE IVP. So converting a second order DE to a first order system is a reversible procedure (when the system arises that way).

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x_1' = x_2$$
$$x_2' = -6x_1 - 7x_2$$
$$x_1(0) = 1$$
$$x_2(0) = 4$$

$$x''(t) + 7x'(t) + 6x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4.$$

So $\quad x_1'' = x_2'$
$$= -6x_1 - 7x_2$$
$$= -6x_1 - 7x_1'$$

So $\quad x_1'' + 7x_1' + 6x_1 = 0$ ✓
$$x_1(0) = 1$$
$$x_1'(0) = x_2(0) = 4$$ ✓

1d) Find the general solution to the first order homogeneous DE in this problem, using the eigendata method we talked about last week. Note the following correspondences, which verify the discussion in the previous parts:

(i) The first component $x_1(t)$ *is* the general solution of the original second order homogeneous DE that we started with.

(ii) The eigenvalue "characteristic equation" for the first order system *is* the same as the "characteristic equation" for the second order DE.

(iii) The "Wronskian matrix" for the first order system *is* a "Wronskian matrix" for the second order DE.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x''(t) + 7x'(t) + 6x(t) = 0$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$

$$= \lambda(7+\lambda) + 6$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$$ ← "same" charac. poly as for DE

evals $\lambda = -1, -6$.

$$E_{\lambda=-1} = Nul \begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad 1 \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \vec{0}$$

$$E_{\lambda=-6} = Nul \begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} = span\left\{ \begin{bmatrix} 1 \\ -6 \end{bmatrix} \right\} \qquad 1 \begin{bmatrix} 6 \\ -6 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}$$

$$\vec{x}_H(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-6t} \\ -c_1 e^{-t} - 6c_2 e^{-6t} \end{bmatrix}$$

for IVP in (b),
Soln was, from above

$$\begin{bmatrix} 2e^{-t} - e^{-6t} \\ -2e^{-t} + 6e^{-6t} \end{bmatrix} = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$= 2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} - 1 \begin{bmatrix} e^{-6t} \\ -6e^{-6t} \end{bmatrix}$$

same as W for 2nd order DE

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \longrightarrow \quad = \begin{bmatrix} e^{-t} & e^{-6t} \\ -e^{-t} & -6e^{-6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$