Note: In analaogy with the scalar undamped oscillator DE

$$x''(t) + \omega_0^2\, x = 0$$

where we could read off and check the solutions

$$\cos\left(\omega_0 t\right),\ \sin\left(\omega_0 t\right)$$

directly without going throught the characteristic polynomial, it is easy to check that

$$\cos(\omega\, t)\underline{\mathbf{v}},\ \sin(\omega\, t)\underline{\mathbf{v}}$$

each solve the conserved energy mass spring system

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$$

as long as

$$-\omega^2\,\underline{\mathbf{v}} = A\,\underline{\mathbf{v}}.$$

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\ .$$

If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are negative, then for each eigenpair $\left(\lambda_j, \underline{\mathbf{v}}_j\right)$ there are two linearly independent solutions to $\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$ given by

$$\mathbf{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\mathbf{v}}_j \qquad \underline{\mathbf{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\mathbf{v}}_j$$

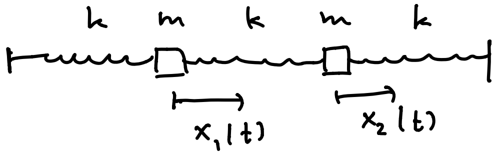
with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

warmup!



$m x_1'' = -k x_1 + k(x_2 - x_1)$
$m x_2'' = -k(x_2 - x_1) - k x_2$

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} .$$

warm-up

$E_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$E_{\lambda=-3} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

b) Eigenvectors same, but evals multiplied by $\frac{k}{m}$

$$\underbrace{\frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\frac{k}{m}\left(-1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$$

$E_{\lambda=-\frac{k}{m}} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $\quad \omega_1 = \sqrt{\frac{k}{m}}$

$E_{\lambda=-3\frac{k}{m}} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ $\quad \omega_2 = \sqrt{\frac{3k}{m}}$

soln

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\underbrace{\left(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_1(t) \equiv x_2(t)}}_{\text{"in-phase" mode}} + \underbrace{\underbrace{\left(c_3 \cos\omega_2 t + c_4 \sin\omega_2 t\right)}_{x_1(t) \equiv -x_2(t)}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{out of phase mode}}$$

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

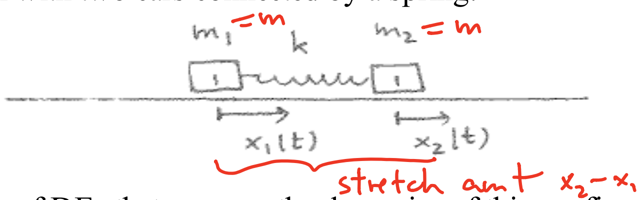
Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

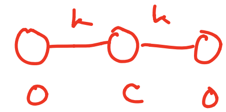
$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 4) Consider a train with two cars connected by a spring:

relates to CO_2 proble





4a) Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

4b). Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}},\ \sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems for next week.

$$\cancel{m}\, x_1''(t) = \frac{k}{m}(x_2 - x_1)$$

$$\cancel{m}\, x_2''(t) = -\frac{k}{m}(x_2 - x_1)$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 1 = (\lambda+1+1)(\lambda+1-1)$$
$$= (\lambda+2)\lambda$$

roots $\lambda = 0, -2$

so for $\frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

eigenvals $\lambda = 0, -\frac{2k}{m}$

continue next week...

Wed Mar 27
5.4 mass-spring systems, and forced oscillations

Announcements:
- practice exam 1–2:20 tomorrow LCB 323
- end of class today – go over topics (you can questions)
- start with warmup exercise.
  review of 2nd order sys → experiment
  finish Tuesday notes.

Warm-up Exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~ e.g. homog tank problem

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~ two mass, three spring problem.

Eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

$$E_{\lambda=-5} = span\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \quad E_{\lambda=-1} = span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

1a) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}\,' = A\vec{x}$ ①
build soltn space out of vector fcns $e^{\lambda t}\vec{v}$
where λ = eval
$A\vec{v} = \lambda\vec{v}$.
try $\vec{x}(t) = e^{\lambda t}\vec{v}$ in ①
$\vec{x}\,'(t) = \lambda e^{\lambda t}\vec{v}$
$A\vec{x} = A e^{\lambda t}\vec{v} = e^{\lambda t} A\vec{v}$
$= e^{\lambda t}\lambda\vec{v}$

1b) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
$+ (c_3 \cos t + c_4 \sin t) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

② $\vec{x}\,'' = A\vec{x}$
$A\vec{v} = \lambda\vec{v} \quad (\lambda < 0)$
$\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$ solve ②, for $\underline{\underline{\omega = \sqrt{-\lambda}}}$ $\quad \omega^2 = -\lambda$

e.g. if $\vec{x}(t) = \cos\omega t\,\vec{v}$
then $\vec{x}\,''(t) = -\omega^2 \cos\omega t\,\vec{v}$ & $A\cos\omega t\,\vec{v} = \cos\omega t\,A\vec{v}$
$= \cos\omega t\,\lambda\vec{v}$
$= -\omega^2\cos\omega t\,\vec{v}$

Math 2280
Experiments
March 27, 2019

We're using the same apparatus as we used for our single mass-spring experiment. In that experiment we measured that an additional mass of 50 g caused the spring to stretch 15.8 cm, and we checked that if we added another 50 g mass, the stretch amount was almost the same - verifying that these springs from Physics have roughly the same Hooke's constant at varying lengths.

So, solving the equation $k\,x = m\,g$ for k we have

> $k := \dfrac{.05 \cdot 9.806}{.158};$

$$k := 3.103164557 \qquad \textbf{(1)}$$

The slow (in-phase) mode has

> $m := .05;$
> $w_1 := \sqrt{\dfrac{k}{m}};$
> $T_1 := \dfrac{2 \cdot \pi}{w_1};$ *#period in seconds*

$$m := 0.05$$
$$w_1 := 7.877816956$$
$$T_1 := 0.7975794998 \qquad \textbf{(2)}$$

20 cycles

17.1
16.9
16.9
17.2
17.1
17.05

$\dfrac{17.05}{20} \approx .853$

in phase

The fast (out of phase) mode has

> $w_2 = \sqrt{3.} \cdot w_1;$
> $T_2 = \dfrac{T_1}{\sqrt{3.}};$ *#period in seconds*

$$12.95105130 = 13.64477922$$
$$T_2 = 0.4604827388 \qquad \textbf{(3)}$$

50 cycles

22.9
23.1
23
23.0

$\dfrac{23}{50} \approx .46$ seconds

!!

out of phase

Probably our experiment will run slow....

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 3.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator back in Chapter 3 notes, assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A-matrix with the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(The "M" matrix turns out to not be diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m := .05;
  ms := .011;
  k := 3.103;
```

$$w_1 := \sqrt{\frac{6 \cdot k}{6 \cdot m + 5 \cdot ms}};$$

$$w_2 := \sqrt{\frac{6 \cdot k}{2 \cdot m + ms}};$$

$$T_1 := evalf\left(\frac{2 \cdot \mathrm{Pi}}{w_1}\right);$$

$$T2 := evalf\left(\frac{2 \cdot \mathrm{Pi}}{w_2}\right);$$

$$m := 0.05$$

$$ms := 0.011$$

$$k := 3.103$$

$$w_1 := 7.241896880$$

$$w_2 := 12.95105130$$

$$T_1 := 0.8676159592 \qquad \text{exp} \quad .853 \text{ sec.}$$

$$T2 := 0.4851486696 \qquad .46 \qquad (4)$$

>

better... I was expecting even better, though ☺

Exam 2 Review
Math 2280-002
Spring 2019

Exam 2 is Friday March 29 from 12:50-1:50 p.m. in our classroom LCB 219. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.

The test will cover sections 3.5-3.6, 4.1, 5.1-5.3, 6.1-6.4, and some of the questions may also tie in to material we covered earlier in the course. Higher likelyhood topics are underlined, and all listed topics are possible unless ruled out explicitly. Percentage weights below add up to more than 100% because exam questions may touch on more than one chapter.

I will post at least one practice test, and will work through it in a problem session Thursday March 28, 2:00-3:20 in a room to be announced.

Chapter 3) 3.5-3.6 (at least 25% of exam)

- 3.5: Finding particular solutions y_p to solve $L(y) = f$, then using the complete solution $y = y_p + y_H$ to solve initial value problems.
    - Undetermined coefficients either in math examples, or in mass-spring oscillation examples from 3.6.

- 3.6 Forced oscillation problems:
    - undamped phenomena: superposition with homogeneous solution, beating, resonance
    - damped phenomena: steady periodic and transient solutions; practical resonance. amplitude-phase form.
    - using conservation of energy TE=PE+KE to derive differential equations of motion for mass-spring and pendulum configuration.

Chapter 4) 4.1 (at least 20% of exam)

- 4.1 Systems of differential equations
    - existence-uniqueness theorem for systems of first order DE's
    - how to convert a second order (or higher order) DE IVP to an equivalent first order system of DE's IVP, and the equivalences between the two frameworks.
    - modeling input-output systems (e.g. tanks) for solute amounts in each compartment

Chapter 5) 5.1-5.5 (at least 40% of the exam)

- 5.1 Theory for linear systems of differential equations:
  - why the solution to $L(y) = f$ is $y = y_P + y_H$
  - why solution space to $L(y) = 0$ is a subspace, and what its dimension is (based on existence-uniqueness theorem).
  - Calculus differentiation rules for sums and products of vector and matrix valued functions.
- 5.2 Eigenvalue-eigenvector method for solving $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$ (Most naturally coupled with an input-output problems or first order system versions of Chapter 3 mass-spring problems).
  - diagonalizable case with real eigenvalues or complex eigenvalues
  - solving $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$ for simple $\underline{\boldsymbol{f}}$, either with $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ or via change of functions using Math 2270 diagonalization identities.
- 5.3 reproducing and classifying phase portraits for $x' = A\,x$ when $n = 2$
  using eigendata and general solutions to sketch phase portraits with real eigenvalues;
  using eigenvalues and sampling tangent field along coordinate axes to sketch portraits in case of complex eigenvalues.

Chapter 6) 6.1-6.4 (at least 25% of the exam)

- 6.1-6.2 Identifying equilibria of first order systems of two autonomous differential equations algebraically. Using linearization and eigendata from Jacobian matrices to classify the type of equilibrium solution, understand the implications for stability, and to be able to sketch what the phase portrait looks like near the equilibrium solution. Interpreting pplane phase portraits.
- 6.3 population models, and what the various terms in the model represent.
- 6.4 nonlinear mechanical models, e.g pendulum and nonlinear springs. Using conservation of energy (or other conserved quantities e.g in section 6.3) in cases where linearization near an equilibrium point is indeterminant, in order to deduce a stable center for the nonlinear problem.