Math 2280-002
Week 11, March 25-27 6.3-6.4, 5.4
Mon Mar 25
6.3-6.4 - further comments about conservative system critical point analysis; 6.5 - the increasing complexity of first order autonomous systems of differential equations as the number of DE's increases, and brief mention of chaotic systems.

Announcements:
- I've posted topic review notes for Friday exam (also at the end of Wed. notes)
- Thursday practice exam session 1:00-2:20 room TBA
- Today "overview" & further directions Chapter 6
  T, W § 5.4 (not on this test).

Warm-up Exercise: (you could look through the notes)

More precise summary of Friday discussions about conserved quantities for first order systems, and resulting saddle points and stable centers for the associated equilbrium points of the first order system. We consider a general first order autonomous system

$$\begin{cases} x'(t) = F(x, y) \\ y'(t) = G(x, y) \end{cases}$$

Theorem: Let $E(x, y)$ be any twice continuously differentiable function with non-degenerate critical points, that is associated to the first order system above in the sense that $E(x(t), y(t))$ is constant along each solution trajectory. (We call $E(x, y)$ a *conserved quantity* for the system of differential equations) Then critical points (x_* , y_*) for the first order systems, i.e. constant solutions, are precisely the points at which $\nabla E = [E_x, E_y] = \mathbf{0}$, i.e. multivariable calculus critical points for $E(x, y)$.

- If $E(x_* , y_*)$ is a local minimum value for E at which the graph $z = E(x, y)$ is concave up (i.e. the Hessian matrix of E is positive definite, i.e. has positive eigenvalues); or if $E(x_* , y_*)$ is a local maximum maximum value for E (i.e the Hessian matrix of E is negative definite, i.e. has negative eigenvalues) - then (x_* , y_*) is a stable center for the first order system of differential equations and the nearby level curves are nearly elliptical.

- If $(x_*, y_*, E(x_* , y_*))$ is a saddle point on the graph, then (x_*, y_*) is a saddle point for the first order nonlinear system.

(One can see why the theorem should be true, but a precise proof would require Math 3220-level analysis. Our two examples from Friday should make the theorem believable though.)

Example 2 from Friday: The second order differential equation for the freely-rotating rigid-rod pendulum,

constant sols.

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

$\sin\theta = 0$

$\theta = n\pi \quad n \in \mathbb{Z}$.

arises from conservation of energy: The total energy

$$TE(t) = KE + PE = \frac{1}{2}mL^2\theta'(t)^2 + mgL(1 - \cos(\theta(t)))$$

equivalent since $x(t) = \theta$, $y'(t) = \theta'$

is constant once the pendulum is set in motion. Thus for the associated system that $[\theta(t), \theta'(t)]$ satisfies, namely

$$x'(t) = y \quad = F(x,y)$$

$$y'(t) = -\frac{g}{L}\sin(x) \quad = G(x,y)$$

$y = 0$

$\sin x = 0$

$x = n\pi, \ n \in \mathbb{Z}$

the (rescaled) energy function

$$E(x,y) = \frac{1}{2}y^2 + \frac{g}{L}(1 - \cos(x)) \quad \frac{TE}{mL^2}$$

$E(x,y)$ critical points

is a conserved quantity. And

$$\nabla E = [E_x, E_y] = \left[\frac{g}{L}\sin(x), y\right] \quad = [0, 0].$$

$\sin x = 0$

$y = 0$ same!!

has exactly the same zeroes ($y = 0$ and $x = n\pi$) as the first order system of differential equations. Furthermore:

Graph of the scaled energy function for rigid-rod pendulum, with $\frac{g}{L} = 1$:

$$E(x, y) = \frac{1}{2} y^2 + \frac{g}{L}(1 - \cos(x))$$

$\nabla E = [0,0] \iff y = 0$, $x = n\pi$

$E(x,y) \geq 0$ minima @ $(2\pi n, 0)$

x even mult of $\cancel{2}\pi$

$(g/L = 1)$

Hessian matrix for E

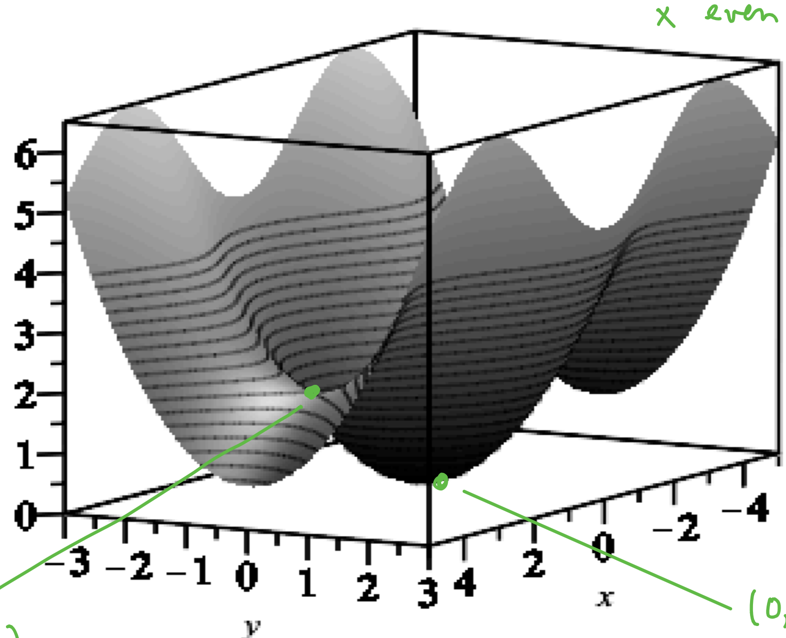
$$\begin{bmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{bmatrix} = \begin{bmatrix} \cos x & 0 \\ 0 & 1 \end{bmatrix}$$

at even mult of π

Hessian is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

local min.

(0,0) Stable center

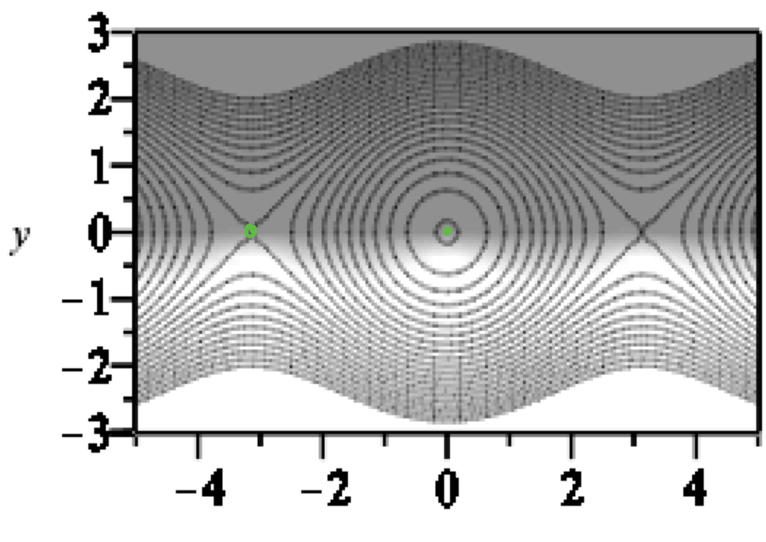


saddle point

$(\pi, 0, 2)$

$(0, 0, 0)$

Top view, showing level curves (contours):

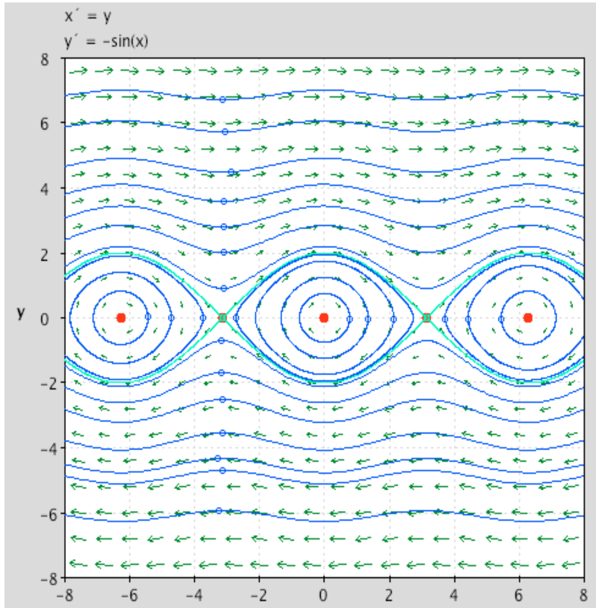


Hessian at every odd mult of π

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

saddle points

pplane output:



Exercise 1 Even though we already know what the answers will be in the undamped case, let's work out the Jacobian matrices and linearizations at the equilibria for the undamped rigid rod pendulum system on the previous pages. Let's consider the possibility of damping at the same time.

even with damping same equil sol's

$$\theta''(t) + c\,\theta'(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

θ const
$\sin\theta = 0$ again
$\theta = n\pi$

know how to do this $\begin{bmatrix} \theta \\ \theta' \end{bmatrix}$

$$x'(t) = y \quad = F$$

$$y'(t) = -\frac{g}{L}\sin(x) - c\,y = G$$

$$\left.\begin{array}{l} y = 0 \\ -\frac{g}{L}\sin x - cy = 0 \end{array}\right\} \begin{array}{l} y = 0 \\ \sin x = 0 \\ x = n\pi. \end{array}$$

$$J_{@(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & -c \end{bmatrix}$$

$\boxed{\begin{array}{l} c = 0 \\ x = n\pi \\ y = 0 \end{array}}$: $J_{@(n\pi,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos n\pi & 0 \end{bmatrix}$

n even, $J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$

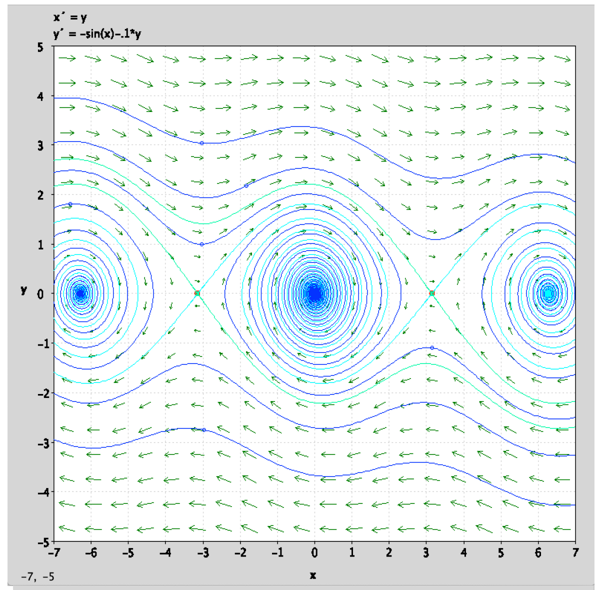
$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{L}$$

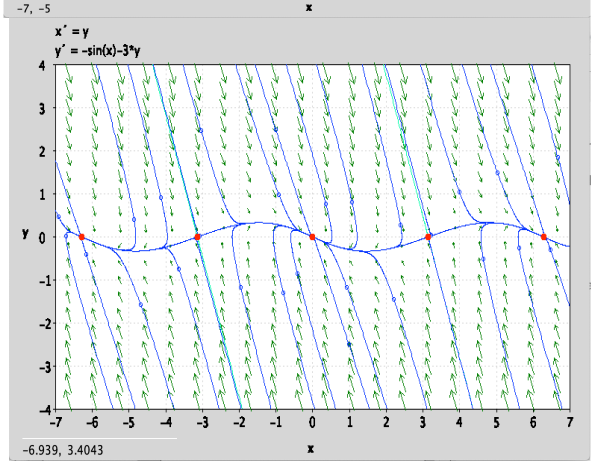
roots $\lambda = \pm i\sqrt{\frac{g}{L}}$

$x = n\pi$ n even
$y = 0$
linearization: stable center.

n odd. $J = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & 0 \end{bmatrix}$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ g/L & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{L}$$ roots $\pm\sqrt{\frac{g}{L}}$ saddle points





with damping

$$J_{@(n\pi,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos n\pi & -c \end{bmatrix}$$

n even $\begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -c-\lambda \end{vmatrix} = \lambda^2 + c\lambda + \frac{g}{L} = 0$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4\frac{g}{L}}}{2}$$

n odd $J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -c \end{bmatrix}$

nodal sink overdamped. $\Big\{$ • $c^2 > 4\frac{g}{L}$ two neg real roots

$$|J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{L} & -c-\lambda \end{vmatrix} = \lambda^2 + c\lambda - \frac{g}{L} = 0 \qquad \lambda = \frac{-c \pm \sqrt{c^2+4\frac{g}{L}}}{2}$$

Saddles

underdamped spiral sinks $\left\{ c^2 < 4\frac{g}{L} \right.$

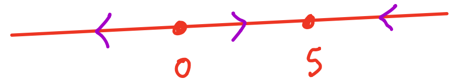
complex roots negative real part

"Show and tell" of possible long-time behavior for solutions to systems of first order autonomous systems, depending on the number of differential equations in the system. We assume the systems satisfy the conditions for the existence-uniqueness theorem. It turns out we've only touched the surface of what can happen, because we've stayed in low dimensions.

$n = 1$. Let $x(t)$ solve a first order autonomous differential equation IVP

$$x'(t) = f(x)$$
$$x(0) = x_0.$$

$x'(t) = 3x(5-x)$



Then either

(1) $\lim_{t\to\infty} x(t) = x_e$, where x_e is an equilibrium solution, or

(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t\to t_1} |x(t)| = \infty$.

$n = 2$. Let $\underline{\mathbf{x}}(t) = [x(t), y(t)]^T$ solve the autonomous system of differential equations IVP

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$
$$x(0) = x_0$$
$$y(0) = y_0$$

Then either

(1) $\lim_{t\to\infty} \underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_e$, where $\underline{\mathbf{x}}_e$ is an equilibrium solution, or

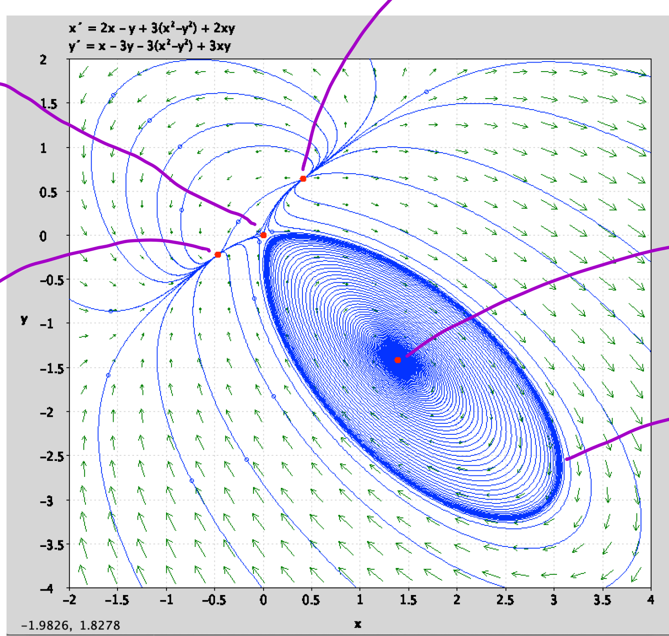
(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t\to t_1} |\underline{\mathbf{x}}(t)| = \infty$.

OR

(3) As $t\to\infty$ $\underline{\mathbf{x}}(t)$ converges to a *periodic limit cycle*. This is the mystery example *pplane* opens with, because it illustrates everything at once.

nodal source

saddle



nodal sink

spiral source

limit cycle (periodic orbit)

$n \geq 3$. Let $\underline{\boldsymbol{x}}(t) \subseteq \mathbb{R}^n$ solve a first order autonomous differential equation IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{f}}(\boldsymbol{x})$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0.$$

Then one of the following happens:

(1) $\lim_{t \to \infty} \underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_e$, where $\underline{\boldsymbol{x}}_e$ is an equilibrium solution, or

(2) there is a $t_1 > 0$ (possibly infinity) so that $\lim_{t \to t_1} \| \underline{\boldsymbol{x}}(t) \| = \infty$

(3) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ converges to a *periodic limit cycle*.

OR

(4) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ "converges" to a *strange attractor*.

(5) As $t \to \infty$ $\underline{\boldsymbol{x}}(t)$ exhibits *chaotic behavior*.

Our text explores some examples of (4),(5) in section 6.5. See also: Math 5410 *Introduction to ordinary differential equations* and Math 5470 *Chaos and non-linear systems*.

Example of strange attractor for the "Lorentz system"
https://en.wikipedia.org/wiki/Lorenz_system

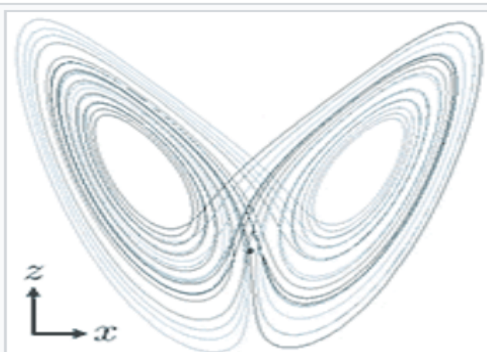
Overview [edit]

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection.[1] -

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$



A sample solution in the Lorenz attractor when ρ = 28, σ = 10, and β = 8/3

Example of period doubling on the way to chaos, for the "forced Duffing equation":
https://en.wikipedia.org/wiki/Duffing_equation

This can be thought of as a (possibly damped) forced oscillation problem, for a mass on top of a wire (see text p. 432):

In your HW 6.4.14

$x'' - 8x + 2x^3 = 0$

$x' = v$

$v' = 8x - 2x^3$

$= 2x(4 - x^2)$

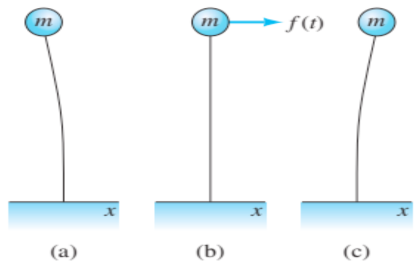


FIGURE 6.5.12. Equilibrium positions of a mass on a filament:
(a) stable equilibrium with $x < 0$;
(b) unstable equilibrium at $x = 0$;
(c) stable equilibrium with $x > 0$.

$$x''(t) + \delta\, x'(t) + \alpha\, x(t) + \beta\, x^3(t) = \gamma \cos(\omega\, t)$$

This can be thought of as an *autonomous* system of three first order DE's, for $[x(t), x'(t),\ t]$, in the time variable e.g. τ:

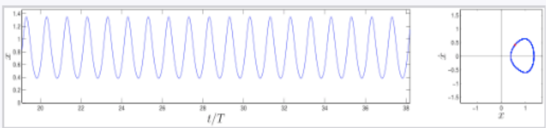
$$x_1'(\tau) = x_2$$
$$x_2'(\tau) = -\alpha\, x_1 - \beta\, x_1^3 - \delta\, x_2 + \gamma \cos(\omega\, \tau)$$
$$t'(\tau) = 1$$

$$x''(t) + \delta x'(t) + \alpha x(t) + \beta x^3(t) = \gamma \cos(\omega t)$$

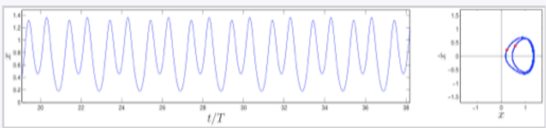
Examples [edit]

Some typical examples of the time series and phase portraits of the Duffing equation, showing the appearance of subharmonics through period-doubling bifurcation – as well chaotic behavior – are shown in the figures below. The forcing amplitude increases from $\gamma = 0.20$ to $\gamma = 0.65$. The other parameters have the values: $\alpha = -1,\ \beta = +1,\ \delta = 0.3$ and $\omega = 1.2$. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$. The red dots in the phase portraits are at times t which are an integer multiple of the period $T = 2\pi/\omega$.[10]

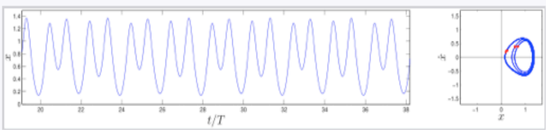
Time traces and phase portraits



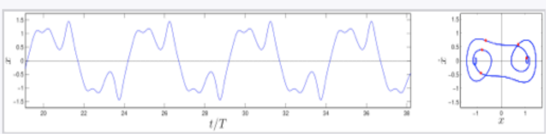
period-1 oscillation at $\gamma = 0.20$



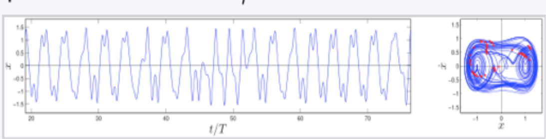
period-2 oscillation at $\gamma = 0.28$



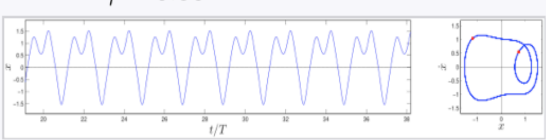
period-4 oscillation at $\gamma = 0.29$



period-5 oscillation at $\gamma = 0.37$



chaos at $\gamma = 0.50$



period-2 oscillation at $\gamma = 0.65$