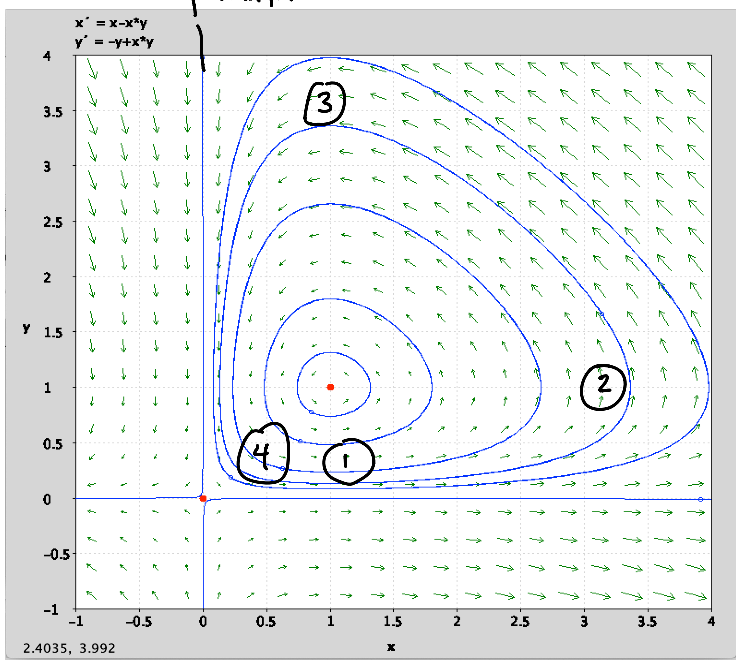
for nonlinear – borderline

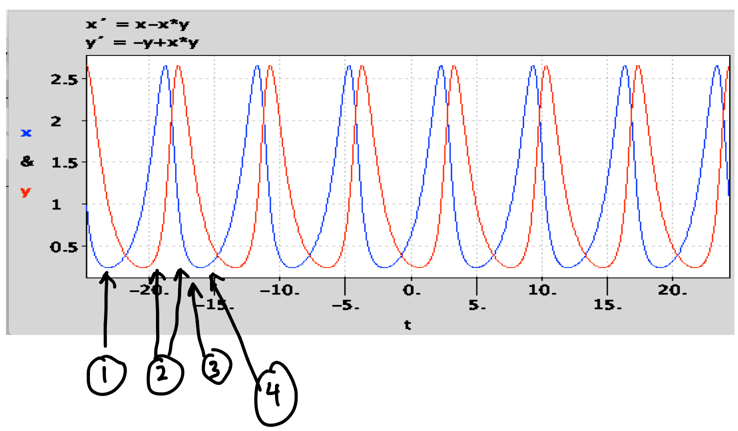
Here's a particular example which shows how the predator-prey system has solutions which oscillate in time. Such behavior can be observed in nature. Depending on time we may do some computations related to this example.

$$\begin{aligned} \text{prey} \quad x'(t) &= x - xy \\ \text{predator} \quad y'(t) &= -y + xy \end{aligned} \qquad \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

really is a stable center

1. few predators $y(t)$; prey $x(t)$ increase.
2. abundant prey $x(t)$ causes predators $y(t)$ to increase, and then, prey $x(t)$ start to decrease
3. prey levels $x(t)$ get so small the predators die off
4. cycle repeats





on the next page we show that the pplane curves are actually the contour lines of an associated conserved function

$$f(x,y) = \ln x - x + \ln y - y.$$

Since $z(t) = t - \ln t$ is concave down and has its max value of -1 at $t=1$,

$f(x,y)$ has its max value of -2 at $(x,y) = (1,1)$.

If (x_0, y_0) is close to $(1,1)$ then it will follow a contour that also stays close to $(1,1)$. This shows $(1,1)$ is a stable center for the non-linear system

These graphics from matlab are related to the phase plane on the previous page! You'll figure out the connection in your homework (see text as well), and we might discuss it in class. Matlab script:

```
%Predator-prey example
X=[.1:.1:4]
Y=[.1:.1:4]
[x,y]=meshgrid(X,Y)
z=log(x)-x +log(y)-y
%This command will plot the graph of z=f(x,y)
% as well as a contour plot below it.
figure
surfc(z)
colormap('gray')
title('predator-prey contour lines')
```

continuing from conclusion below, note that

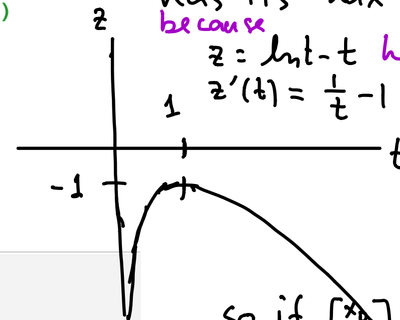
$f(x,y) = \ln x - x + \ln y - y$

has its max at $(1,1)$

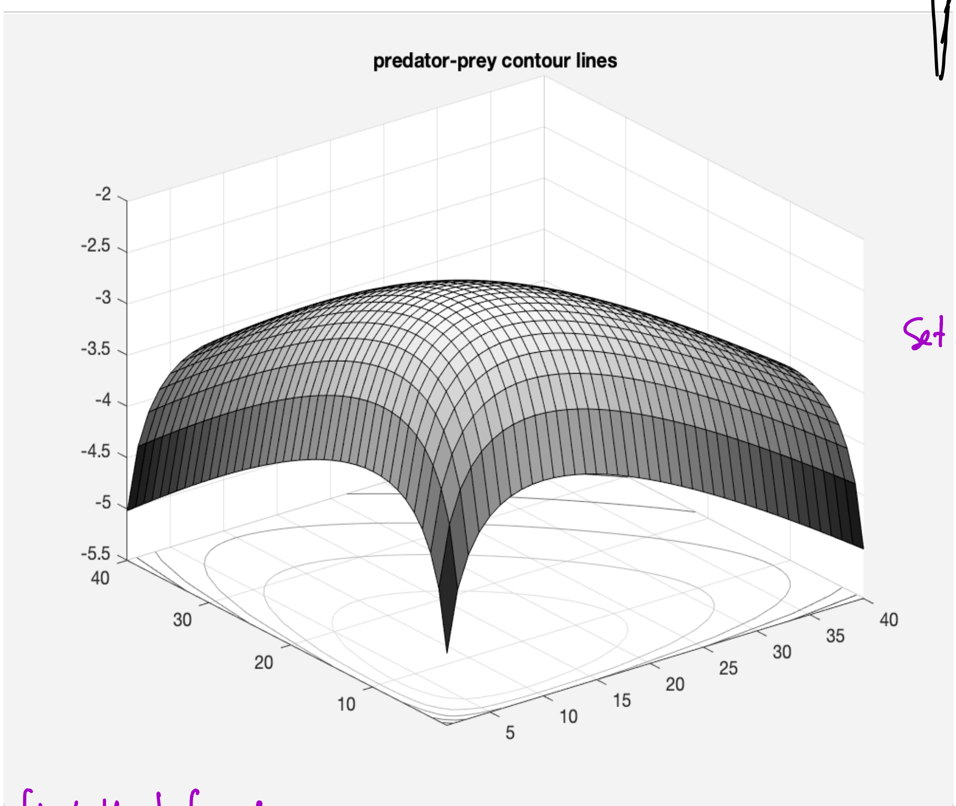
because

$z = \ln t - t$ has max @ $t=1$

$z'(t) = \frac{1}{t} - 1 = 0$ at $t=1$



output:



so if $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is close to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $f(x_0,y_0)$ is close to its max so $[x(t), y(t)]$ stays close to $(1,1)$. Because

Set up: predator-prey curves follow contours of a function which has a strict max at equil pt, (and is "strictly" concave down)

How to find that fcn??

start here to find that fcn:

$$x'(t) = x - xy$$
$$y'(t) = -y + xy$$

so $\longrightarrow$ $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

$$\frac{dy}{dx} = \frac{-y+xy}{x-xy}$$

$$\frac{dy}{dx} = \frac{y(-1+x)}{x(1-y)}$$

$$\frac{1-y}{y}\,dy = \int \frac{-1+x}{x}\,dx$$

$$\int \frac{1}{y} - 1\,dy = \int -\frac{1}{x} + 1\,dx$$

$$\ln y - y = -\ln x + x + C$$

$$\boxed{\ln y - y + \ln x - x = C}$$

Solution trajectories $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ follow level curves of $f(x,y) = \ln y - y + \ln x - x$

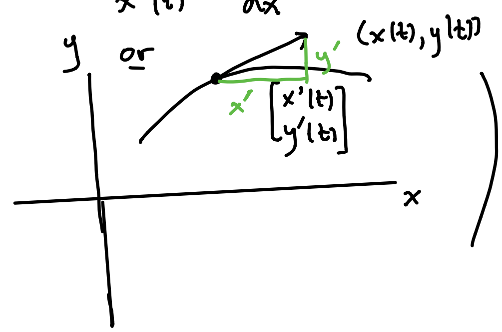
conclusion

we found that fcn!!

$\left(\frac{d}{dt} y(x(t)) = \frac{dy}{dx}\frac{dx}{dt} \right.$

$\frac{y'(t)}{x'(t)} = \frac{dy}{dx}$

or



$\left. \right)$

Fri Mar 22
6.4 Nonlinear mechanical systems

Announcements:
- Pick up applications handout & completed hw assignment
  w10.1
  w10.2 today
- Wed notes finish first.

Warm-up Exercise: we'll talk about the application handout - you could look it over

Math 2280-002   Friday ~~January~~ March 22 Supplement.

We discussed a simplified linear Glucose-insulin model before break, and more recently we've been discussing nonlinear interactions between two "populations". Mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more comprehensive) models in the same vein.

Prof. Fred Adler - who has a joint appointment in Math and Biology, worked with a Math graduate student Chris Remien and collaborated with University Hospital around 7 years ago to model liver poisoning by acetominophen (brand name Tyleonol). Accidental overdoses of that particular painkiller are a leading cause of liver failure. They studied a non-linear system of 8 first order differential equations, and came up with a new diagnostic test which can be useful in deciding the extent and time frame of the overdose, in patients who are brought to the hospital unconscious. This is decisive information in whether the antidote will still save their liver, or whether they need a liver transplant to survive.

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Math Can Save Tylenol Overdose Patients

NEW WAY FOR DOCS TO PREDICT WHO NEEDS LIVER TRANSPLANTS

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Feb. 27, 2012 – University of Utah mathematicians developed a set of calculus equations to make it easier for doctors to save Tylenol overdose patients by quickly estimating how much painkiller they took, when they consumed it and whether they will require a liver transplant to survive.

"It's an opportunity to use mathematical methods to improve medical practice and save lives," says Fred Adler, a professor of mathematics and biology and coauthor of a study that developed and tested the new method.

The study of acetaminophen – the generic pain and fever medicine sold as Tylenol and in many other nonprescription and prescription drugs – was set for publication within a week in *Hepatology*, a journal about liver function and disease.

Adler, math doctoral student Chris Remien and their colleagues showed that using only four common medical lab tests – known as AST, ALT, INR and creatinine – the equations can quickly and accurately predict which Tylenol overdose patients will survive with medical treatment and which will die unless they receive a liver transplant.

The researchers analyzed the records of 53 acetaminophen overdose patients treated at the University of Utah's University Hospital to test the equations and show they quickly and accurately predicted, in retrospect, which patients survived and which died.

Speed is essential in listing acute liver failure patients as candidates for transplant, says study coauthor Norman Sussman, a former University of Utah liver doctor now at the Baylor College of Medicine in Houston.

There's a link to their published paper in the previous link, and it would also be easy to find on google scholar. For fun, I copied and pasted the non-linear system of first order differential equations that they study. Notice the quadratic interaction terms for how the various quanitites interact. This is typical in these sorts of models, because chemical reactions typically happen at rates proportional to the amounts of reactants in the system.

Partial key:

APAP = Acetaminophen

NAPQ1 = one of the toxins

GSH = Gluthione which binds to the toxin and eliminates it

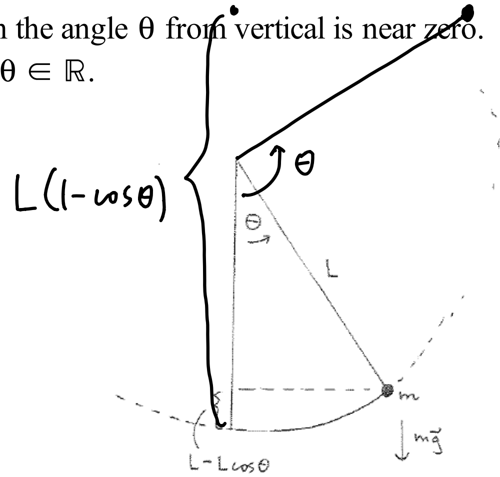
AST, ALT = enzymes released by damaged hepatocytes, which have known typical clearance time curves, and which are measured upon admission, along with the "prothrombin time/international normalized ratio (INR)" clotting factor, and then the systems is solved in the negative time direction to deduce approximate time and amount of APAP ingestion.

http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

APAP	$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$
NAPQI	$\frac{dN}{dt} = \frac{q p \alpha}{H_{max}} A - \gamma N G$
GSH	$\frac{dG}{dt} = \kappa - \gamma N G - \delta_g G$
Functional Hepatocytes	$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta N H$
Damaged Hepatocytes	$\frac{dZ}{dt} = \eta N H - \delta_z Z$
AST	$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$
ALT	$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$
Clotting Factor	$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F\right)$

Example 1) The rigid rod pendulum.

We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.



In Chapter 3 we used conservation of energy to derive the dynamics for this (now) swinging, or possibly rotating, pendulum. There were no assumptions about the values of θ in that derivation of the non-linear DE (it was only when we linearized that we assumed θ was near zero). We began with the total energy

$$TE = KE + PE = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m\left(L\,\theta'(t)\right)^2 + m\,g\,L\left(1 - \cos(\theta(t))\right)$$

And set $TE'(t) \equiv 0$ to arrive at the differential equation

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0 \quad .$$

$\theta'' + \frac{g}{L}\sin\theta = 0$ $\qquad$ $\frac{d}{dt}\begin{bmatrix}\theta \\ \theta'\end{bmatrix} = \begin{bmatrix}\theta' \\ -\frac{g}{L}\sin\theta\end{bmatrix}$

We see that the constant solutions $\theta(t) = \theta_*$ must satisfy $\sin\left(\theta_*\right) = 0$, i.e. $\theta_* = n\,\pi, \pi \in \mathbb{R}$. In other words, the mass can be at rest at the lowest possible point (if θ is an even multiple of π), but also at the highest possible point (if θ is any odd multiple of π). We expect the lowest point configuration to be a "stable" constant solution, and the other one to be "unstable".

We will study these stability questions systematically using the equivalent first order system for

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$

when $\theta(t)$ represents solutions the pendulum problem. You can quickly check that this is the system

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x).$$

Notice that constant solutions of this system, $x' \equiv 0, y' \equiv 0$, equivalently

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_* \\ y_* \end{bmatrix} \text{ equals constant}$$

must satisfy $y_* = 0$, $\sin(x_*) = 0$, In other words, $x = n\,\pi$, $y = 0$ are the equilibrium solutions. These correspond to the constant solutions of the second order pendulum differential equation, $\theta = n\,\pi$, $\theta' = 0$.

Here's a phase portrait for the first order pendulum system, with $\frac{g}{L} = 1$, see below.

a) Locate the equilibrium points on the picture and verify algebraically.

b) Interpret the solution trajectories in terms of pendulum motion.

c) Looking near each equilibrium point, and recalling our classifications of the origin for linear homogenous systems (spiral source, spiral sink, nodal source, nodal sink, saddle, stable center), how would you classify these equilbrium points and characterize their stability?

equil solns were

$x = n\pi \quad n \in \mathbb{Z} \quad (x(t) = \theta(t))$

$y = 0 \quad (y(t) = \theta'(t))$

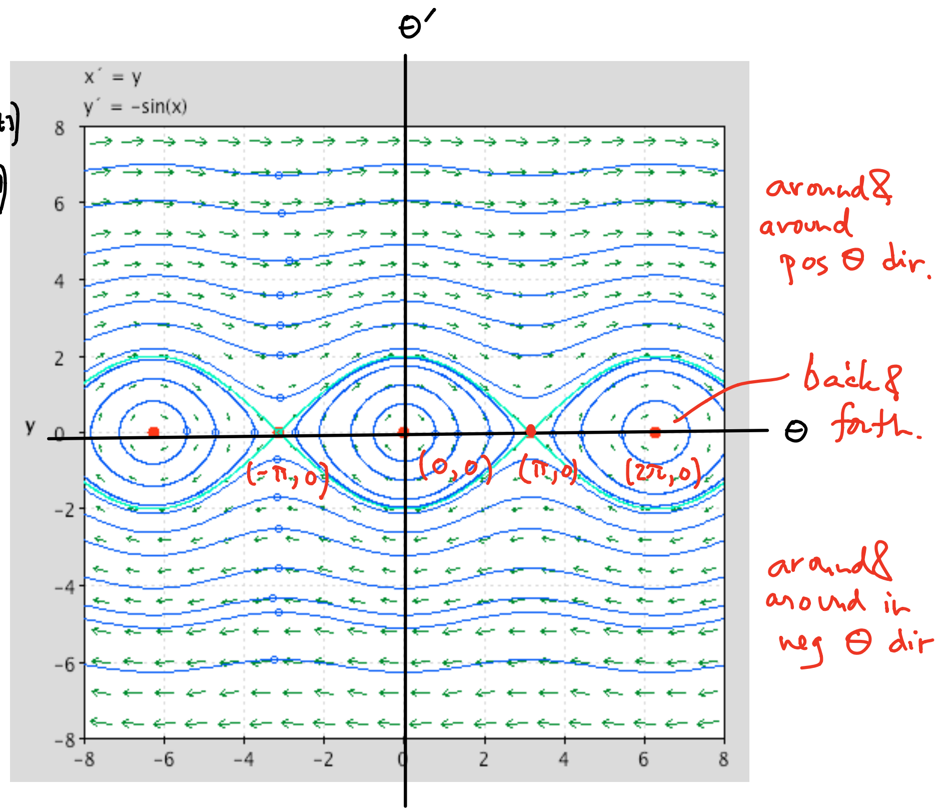
$x = n\pi \quad n$ even

looks like center

$x = n\pi \quad n$ odd

saddle

really are stable centers



$$TE = \frac{1}{2}mv^2 + mgL(1-\cos\theta) \longrightarrow TE(x,y) = \frac{1}{2}mL^2y^2 + mgL(1-\cos x)$$

is constant for solution trajectories

$x = \theta(t)$

$y = \theta'(t)$

$v = L\theta'(t)$

So for $\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} 2\pi n \\ 0 \end{bmatrix}$

$TE = 0$, smallest possible

what y value makes PE as small as possible $y = 0$

what x values make this as small as possible $x = 2\pi n$, $n \in \mathbb{Z}$

If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is close to $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ its TE is close to 0

so $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ stays close to $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ stable center

Exercise 1 Work out the Jacobian matrices and linearizations at the equilibria for the undamped rigid rod pendulum system on the previous page

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x)$$

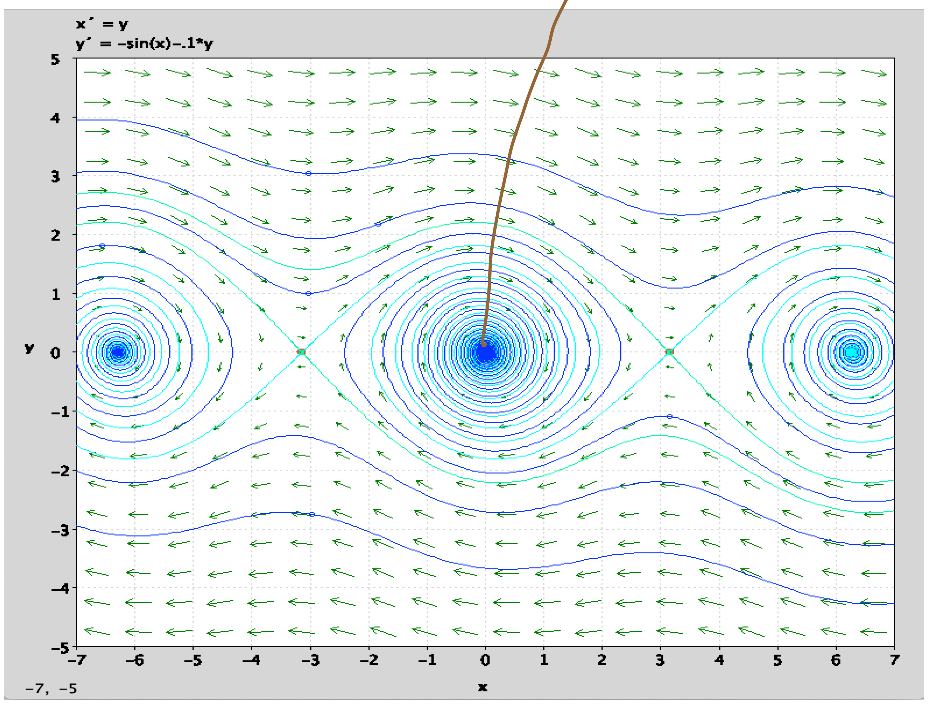
Exercise 2 What happens when you add damping?

$$\theta''(t) + c\,\theta'(t) + \frac{g}{L}\sin(\theta(t)) = 0$$

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x) - c\,y$$

spiral sinks

c = .1



c=3
overdamped