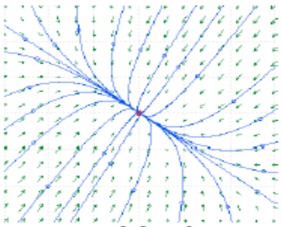
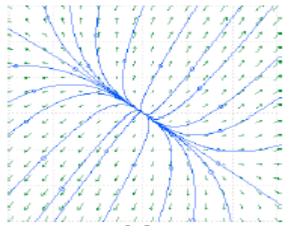
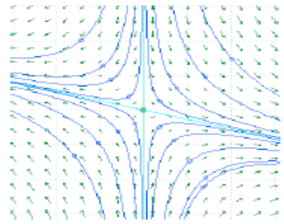
The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



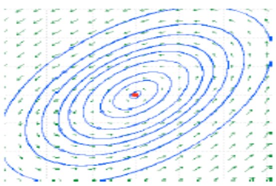
nodal sink
$\lambda_1, \lambda_2 < 0$



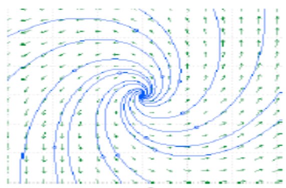
nodal source
$\lambda_1, \lambda_2 > 0$



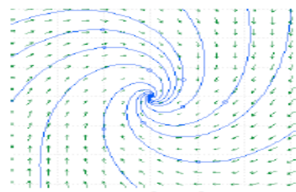
saddle point
$\lambda_1 < 0 < \lambda_2$



center
$\mathrm{Re}(\lambda)=0$



spiral source
$\mathrm{Re}(\lambda)>0$

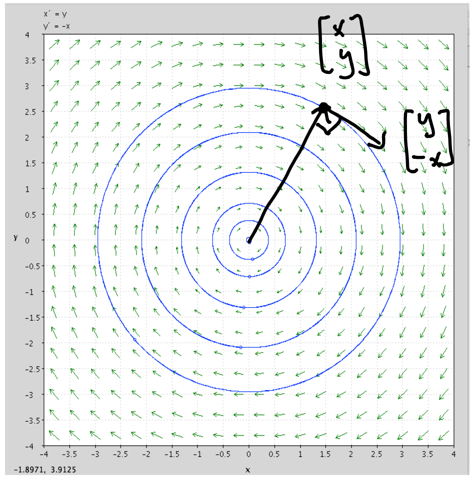


spiral sink
$\mathrm{Re}(\lambda)<0$

<u>Theorem:</u> Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.
(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.
(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Fun examples of borderline cases where the linearization at the origin has purely imaginary eigenvalues, so the origin is a stable center for the linearization but all three flavors for the three nonlinear systems:

stable center



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

note $x'' = y' = x$

$x'' + x = 0$

$p(r) = r^2 + 1$

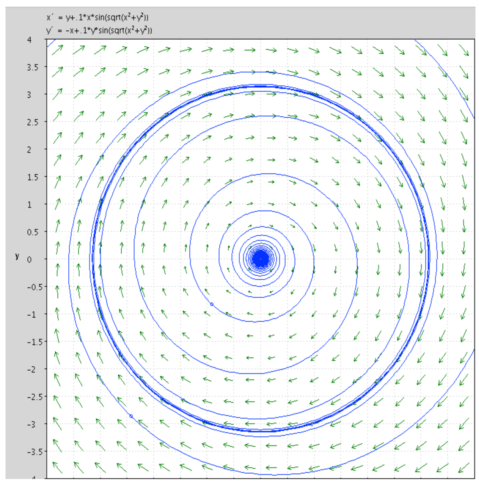
$x(t) = A\cos(t-\alpha)$

$x'(t) = y = -A\sin(t-\alpha)$

orbits circles.

$|A - \lambda I| = \lambda^2 + 1$

roots $\lambda = \pm i$



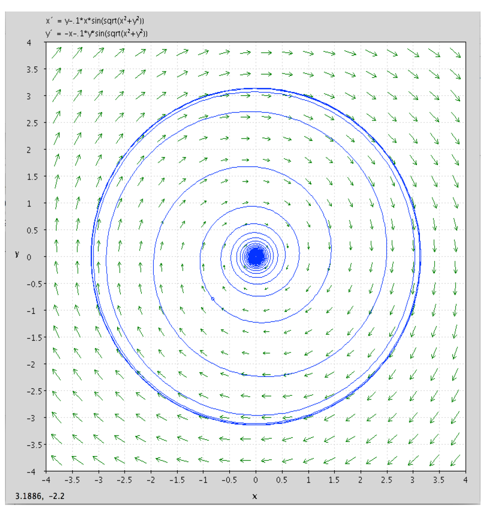
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} + \underbrace{.1\sin\sqrt{x^2+y^2}}_{\text{positive}} \begin{bmatrix} x \\ y \end{bmatrix}$$

as before

$0 < \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| < \pi$

outward radial

results in spiral source because of higher order



Wed Mar 20
6.3 Ecological models, continued

Announcements:

- add to Friday notes, & some HW.
- for HW, we'll finish by Friday.
  next week back to Chapter 5
  (new HW assignment → for Wed after exam)

Warm-up Exercise:

This is a "predator-prey" model. Which is which? Also, what are the equilibrium solutions?

$$x'(t) = ax - pxy$$
$$y'(t) = -by + qxy$$

$a, b, p, q > 0.$

prey: $-pxy$ says $x(t)$ doesn't benefit from $y(t)$
$+qxy$ says y' is greater if there is more x

ax if no y, then $x' = ax$ → grows exponentially
$-by$ if there's no x, y dies off exponentially.

So $x(t)$ is the prey
$y(t)$ is the predator.

There are many interesting two-species models. In class and in one of your homework problems we've considered examples of the *logistic competition model* between two species:

logistic | competition

$$x'(t) = a_1\, x - b_1\, x^2 - c_1 x\, y$$
$$y'(t) = a_1\, y - b_2\, y^2 - c_2\, x\, y$$

Here the constants $a_1, a_2, b_1, b_2, c_1, c_2$ are all positive. It turns out that if the logistic inhibition, as measured by the product $b_1 b_2$ is stronger than the competitive pressure as measured by $c_1 c_2$, i.e.

$$b_1 b_2 > c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always asymptotically stable. This is what happened in our class example

$$x'(t) = 14\, x - 2\, x^2 - x\, y$$
$$y'(t) = 16\, y - 2\, y^2 - x\, y\ .$$

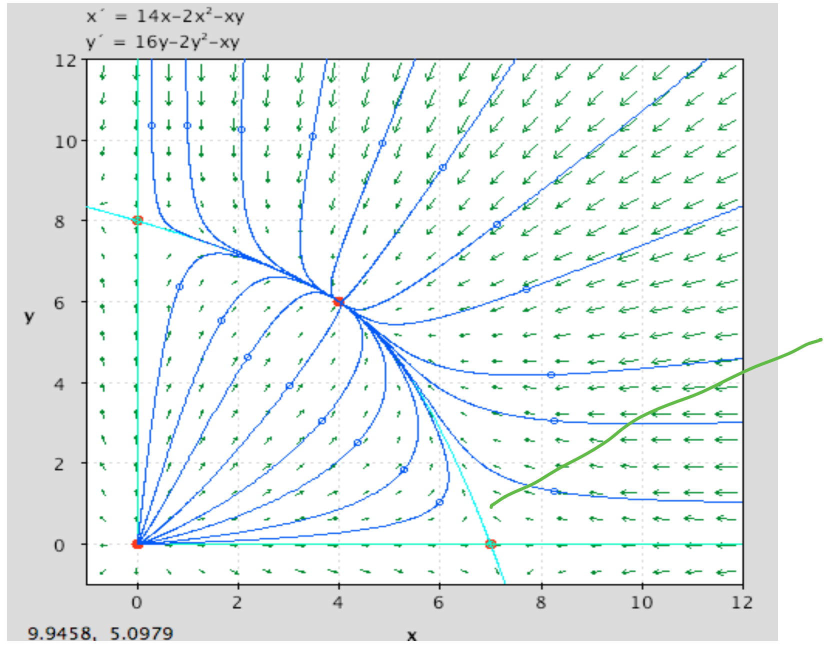
On the other hand, if

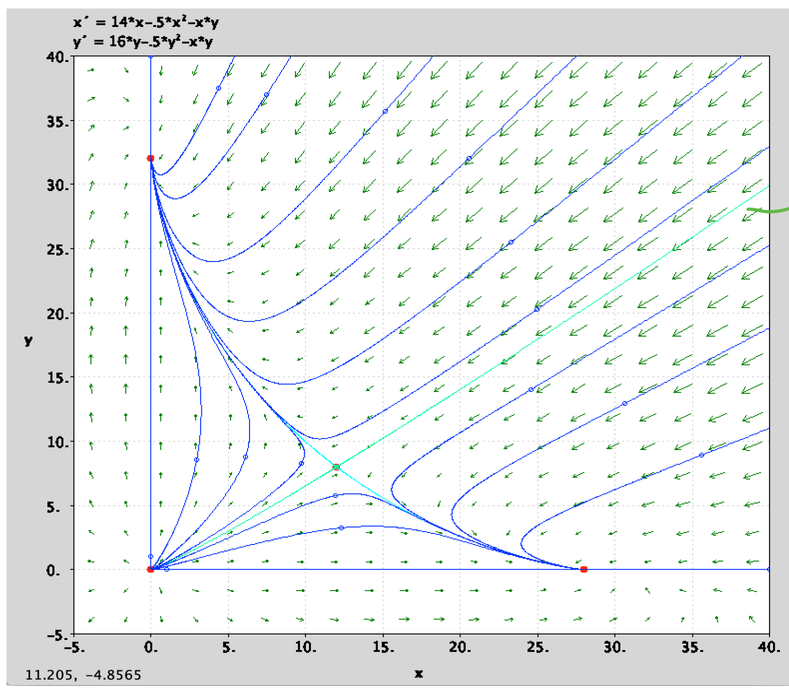
$$b_1 b_2 < c_1 c_2$$

and if there is a first quadrant equilbrium solution (x_*, y_*) then it is always unstable! This is what happened in your homework problem

$$x'(t) = 14\, x - .5\, x^2 - x\, y$$
$$y'(t) = 16\, y - .5\, y^2 - x\, y$$

pictures on next page...





seperatrix
or
stable
orbit for
$\begin{bmatrix} 12 \\ 8 \end{bmatrix}$

notice
how it
divides
the long
term
limits
based
on $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

Another model is the classical predator prey model, for prey $x(t)$ and predator $y(t)$. All constants are positive:

$$x'(t) = a\,x - p\,x\,y = x\,(a - p\,y) = F(x,y)$$
$$y'(t) = -b\,y + q\,x\,y = y(-b + q\,x) := G(x,y)$$

See warmup

Exercise 1

a) Find the ~~4~~ 2 equilibrium solutions

constant

$$F(x,y) = 0 = x(a-py)$$
$$G(x,y) = 0 = y(-b+qx)$$

So $x=0$ or $a-py=0$

If $x=0$ then $y=0 \longrightarrow \begin{bmatrix}0\\0\end{bmatrix}$

or $-b+qx=0 \rightarrow$ inconsistent no sol'n.

If $a-py=0$ $y=\frac{a}{p}$

then $y=0 \longrightarrow$ inconsistent

or $-b+qx=0$ $x=\frac{b}{q}$

$\longleftarrow \begin{bmatrix}a/p\\b/q\end{bmatrix}$

b) The most interesting equilibrium solution is the one in the first quadrant,

$$(x_E, y_E) = \left(\frac{b}{q}, \frac{a}{p}\right).$$

Show that the linearization at this equilibrium point always yields a stable center, which is the borderline case. So, this equilibrium is indeterminant for the nonlinear system. It turns out that for the nonlinear system, however, this first quadrant equilbrium solution *is* always a stable center. You'll explore these ideas further in homework...

$$x(t) = x_E + u(t)$$
$$y(t) = y_E + v(t)$$

$u(t), v(t)$ measure deviations from x_E y_E

$$x'(t) = ax - pxy = F$$
$$y'(t) = -by + qxy = G$$

$\begin{bmatrix}u(t)\\v(t)\end{bmatrix}$ approx sats $\begin{bmatrix}u'(t)\\v'(t)\end{bmatrix} = A\begin{bmatrix}u\\v\end{bmatrix}$

$A = J_{@\begin{bmatrix}x_E\\y_E\end{bmatrix}}$

$$J_{@(x,y)} = \begin{bmatrix}F_x & F_y\\ G_x & G_y\end{bmatrix} = \begin{bmatrix}a-py & -px\\ qy & -b+qx\end{bmatrix}$$

$$J_{@\left(\frac{b}{q},\frac{a}{p}\right)} = \begin{bmatrix}a - p\frac{a}{p} & -p\frac{b}{q}\\ q\frac{a}{p} & -b+q\frac{b}{q}\end{bmatrix} = \begin{bmatrix}0 & -p\frac{b}{q}\\ q\frac{a}{p} & 0\end{bmatrix}$$

$$|J-\lambda I| = \begin{vmatrix}-\lambda & -\frac{pb}{q}\\ \frac{qa}{p} & -\lambda\end{vmatrix} = \lambda^2 + \frac{\cancel{q}a}{\cancel{p}}\frac{\cancel{p}b}{\cancel{q}} = \lambda^2 + ab$$

roots $\lambda = \pm i\sqrt{ab}$

for linearized, (stable) center

for nonlinear – borderline

Here's a particular example which shows how the predator-prey system has solutions which oscillate in time. Such behavior can be observed in nature. Depending on time we may do some computations related to this example.

$$\begin{array}{ll} \text{prey} & x'(t) = x - xy \\ \text{predator} & y'(t) = -y + xy \end{array} \qquad \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

really is a stable center

1. few predators $y(t)$; prey $x(t)$ increase.
2. abundant prey $x(t)$ causes predators $y(t)$ to increase, and then, prey $x(t)$ start to decrease
3. prey levels $x(t)$ get so small the predators die off
4. cycle repeats