Introduction to Chapter 6. (We'll return to Chapter 5 after we discuss Chapter 6.) This chapter is about general (non-linear) systems of two first order differential equations for $x(t), y(t)$, i.e.

$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$

which we often abbreviate, by writing

$$x' = F(x, y, t)$$
$$y' = G(x, y, t) \ .$$

If we assume further that the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t, i. e.

$$x' = F(x, y)$$
$$y' = G(x, y)$$

then we call such a system autonomous. Autonomous systems of first order DEs are the focus of most of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2.

Constant solutions to an autonomous differential equation or system of DEs are called equilibrium solutions. Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system

$$F(x, y) = 0$$
$$G(x, y) = 0$$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems

$$x' = F(x, y)$$
$$y' = G(x, y)$$

are called stable if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.

- Equilibrium solutions are unstable if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called asymptotically stable if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.

(Notice these definitions are completely analogous to our discussion in Chapter 2.)

stable: $\forall \, \varepsilon > 0$ (that's how close you to stay) $\exists \, \delta > 0$ (how close you need to start) s.t. $\left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} x_* \\ y_* \end{bmatrix} \right\| < \delta$

then $\left\| \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} x_* \\ y_* \end{bmatrix} \right\| < \varepsilon$

for all $t > 0$.

Exercise 4) Consider the "competing species" model from section 6.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

logistic   competition

$$x'(t) = 14\,x - 2\,x^2 - x\,y \quad = F$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y\,. \quad = G$$

a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other). Hint: to understand why this model is plausible for $x(t)$ consider the normalized birth rate $\dfrac{x'(t)}{x(t)}$, as we did in Chapter 2.

b) $14x - 2x^2 - xy = 0 = x(14 - 2x - y)$
$16y - 2y^2 - xy = 0 = y(16 - 2y - x)$

b) Find the four equilibrium solutions to this competition model, algebraically.
c) What does the phase portrait below indicate about the dynamics of this system?
d) Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability, based on what the phase portrait looks like near each equilibrium solution?

$x = 0$ or $14 - 2x - y = 0$

and $y = 0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and $16 - 2y - x = 0$: $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$

and $y = 0$: $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

and $16 - 2y - x = 0$: $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$2x + y = 14$
$x + 2y = 16$
solve!

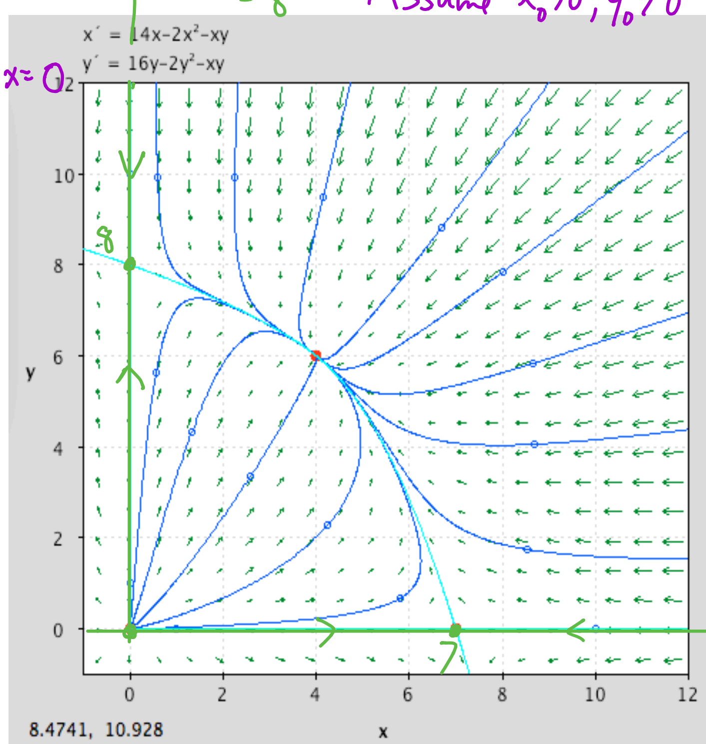
if $x = 0$
$y' = 16y - 2y^2 = 2y(8 - y)$
logistic
carry capac 8

along y-axis, get logistic phase diagram from Chapter 2, for $y(t)$

(c)

Assume $x_0 > 0, y_0 > 0$

appears $\lim_{t\to\infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$



if $y = 0$,
$x' = 14x - 2x^2$
$= 2x(7 - x)$
get logistic phase diagram for $x(t)$, Chptr 2

Tues Mar 19
6.1-6.2 Autonomous systems of two first order differential equations; linearization near equilibrium solutions.

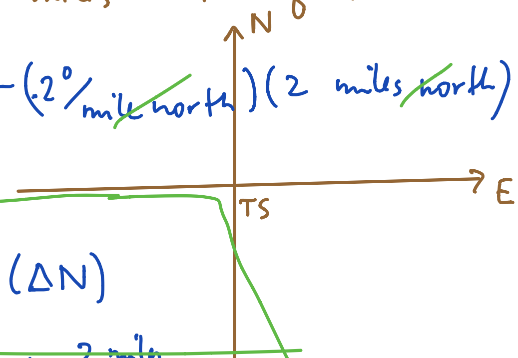
Announcements:

- for linearization, need to recall key idea from differential multivariable Calc

  here!

- Monday – Tuesday Chapter 6

Warm-up Exercise: Suppose the temperature downtown (say at Temple Square) is currently 45°. And suppose that nearby, it decreases about .2°/mile as you go north, and .5°/mile as you go east.
Approximate the temperature at the airport, which is about 8 miles west and 2 miles north of T.S.



$$T(A.P.) \approx 45^\circ + \left(.5^\circ/\cancel{\text{mile}}\ \cancel{\text{west}}\right)\left(8\ \cancel{\text{miles}}\ \text{west}\right) - \left(.2^\circ/\cancel{\text{mile}}\ \cancel{\text{north}}\right)\left(2\ \text{miles}\ \cancel{\text{north}}\right)$$

$$= 45 + 4 - .4$$
$$= 48.6^\circ$$

$$\boxed{T(A.P.) \approx T(T.S.) + \underbrace{\frac{\partial T}{\partial E}(T.S.)}_{-.5^\circ/\text{mile}}\underbrace{(\Delta E)}_{(-8\ \text{miles})} + \underbrace{\frac{\partial T}{\partial N}}_{-.2^\circ/\text{mile}}\underbrace{(\Delta N)}_{2\ \text{miles}}}$$

Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math 2280 course. (The "linear drag" velocity model, Newton's law of cooling, small oscillation pendulum motion, and the damped spring equation were all linearizations.) It's important to understand how to linearize in general, because linearized differential equations can often be used to understand stability and solution behavior near equilibrium points, for the original differential equations. Today we'll talk about linearizing *systems* of DE's, which we've not done before in this course.

An easy case of linearization in Exercise 4 from Monday's notes is near the equilbrium solution $[x_*, y_*]^T = [0, 0]^T$. It's pretty clear that our rabbit-squirrel population system

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

linearizes to

$$x'(t) = 14x$$
$$y'(t) = 16y$$

$x(t) = c_1 e^{14t}$
$y(t) = c_2 e^{16t}$

i.e.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

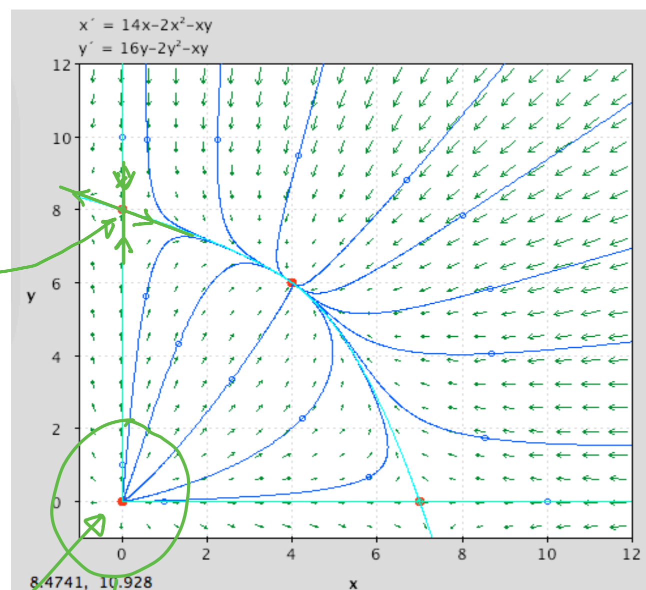
The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:

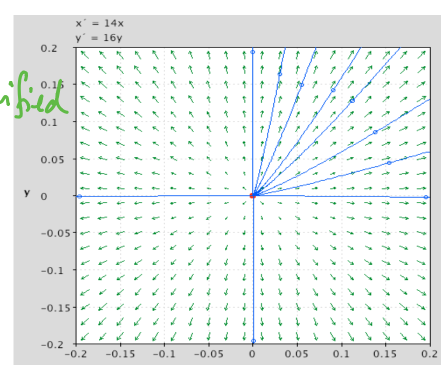
nodal source,

saddle



nodal source

like magnified

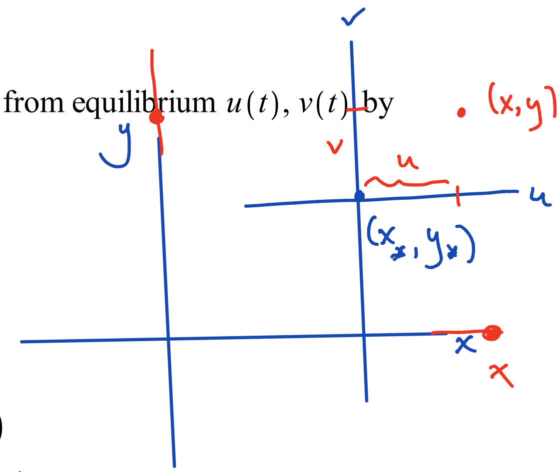


We use multivariable Calculus to linearize at equilibrium points that are not the origin. (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter.)
Here's how: Consider the autonomous system

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \quad .$$



For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \ .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

warmup

$$u' = F(x_* + u, y_* + v) = \cancelto{0}{F(x_*, y_*)} + \frac{\partial F}{\partial x}(x_*, y_*)\, u + \frac{\partial F}{\partial y}(x_*, y_*)\, v + \varepsilon_1 \cancel{(u, v)}$$

$$v' = G(x_* + u, y_* + v) = \cancelto{0}{G(x_*, y_*)} + \frac{\partial G}{\partial x}(x_*, y_*)\, u + \frac{\partial G}{\partial y}(x_*, y_*)\, v + \varepsilon_2 \cancel{(u, v)}$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$.
Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

The matrix of partial derivatives is called the Jacobian matrix for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Exercise 1) We will linearize the rabbit-squirrel (competition) model of the running example, near the equilibrium solution $[4,6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy = F(x,y)$$
$$y'(t) = 16y - 2y^2 - xy = G(x,y)$$

1a) Use the Jacobian matrix method of linearizing they system at $[4,6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

$$= \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \checkmark$$

$$J_{@(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 14-4x-y & -x \\ -y & 16-4y-x \end{bmatrix}. \quad J_{@(4,6)} = \begin{bmatrix} 14-16-6 & -4 \\ -6 & 16-24-4 \end{bmatrix}$$

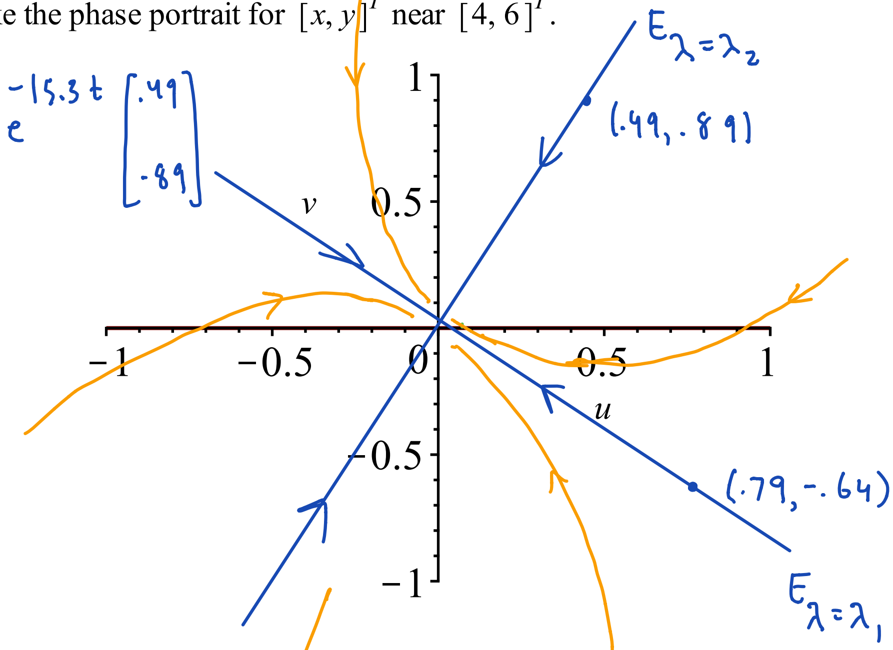
1b) The matrix in the linear system of DE's above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{\boldsymbol{v}}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{\boldsymbol{v}}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u,v]^T = [0,0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, the phase portrait for the linearized problem near $[u,v]^T = [0,0]^T$ looks very much like the phase portrait for $[x,y]^T$ near $[4,6]^T$.

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \approx c_1 e^{-4.7t} \begin{bmatrix} .79 \\ -.64 \end{bmatrix} + c_2 e^{-15.3t} \begin{bmatrix} .49 \\ .89 \end{bmatrix}$$

nodal sink,
asymptotically stable

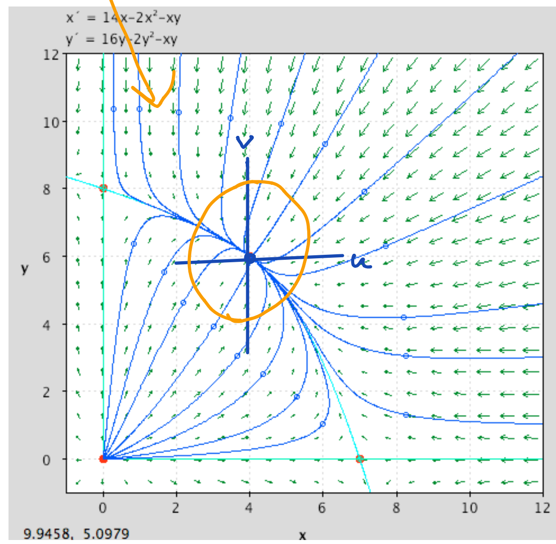


note all other orange curves are dilations of these

compare

Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

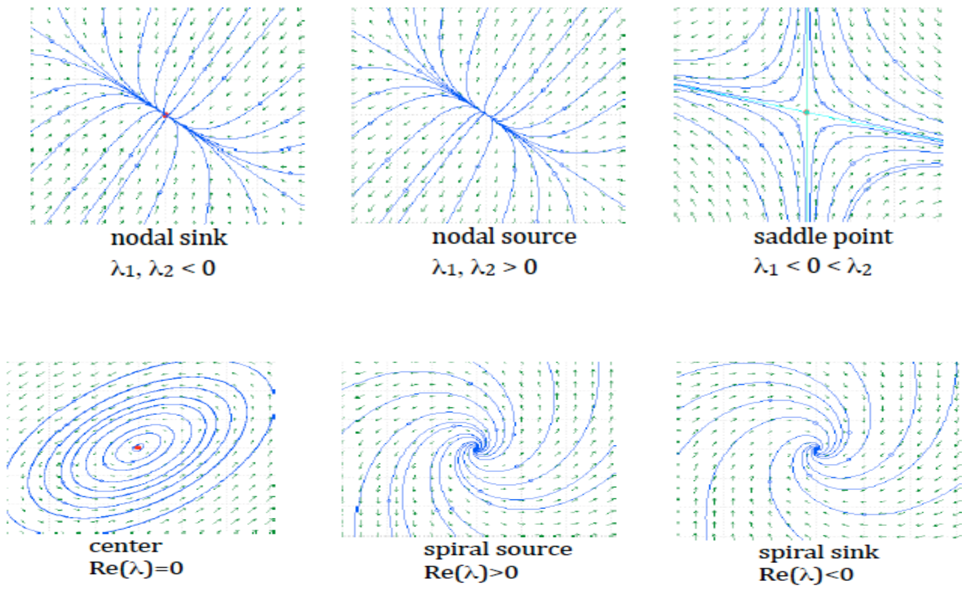
The non-linear problem and representative solution curves:



pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

```
Equilbrium Point:
    There is a nodal sink at (4, 6)
    Jacobian:
            -8        -4
            -6        -12
    The eigenvalues and eigenvectors are:
            -4.7085    (0.77218, -0.63541)
            -15.292    (0.48097, 0.87674)
```

The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



<u>Theorem:</u> Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.
(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.
(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Fun examples of borderline cases where the linearization at the origin has purely imaginary eigenvalues, so the origin is a stable center for the linearization but all three flavors for the three nonlinear systems: