Math 2280-002
Week 10, March 18-22 5.3, 6.1-6.4
Mon Mar 18
5.3 - phase portraits for homogeneous systems of two linear DE's - summary of complex eigendata case from Friday, and discussion of real eigendata examples; 6.1 Introduction to systems of two autonomous first order differential equations.

Announcements: Exam 2 next Friday

take on time, understand shapes of phase portraits as relates to eigendata of $A_{2\times 2}$

Warm-up Exercise: no warmup today, but if you missed Friday maybe scan page 3 of today's notes → we understood $n=2$ homogeneous phase portraits with $\mathbb{C}$ eigendata.

5.3 phase portraits for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A. This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points. On Friday before break we analyzed the case of complex eigendata. Today we'll analyze what happens with real eigendata.

Here's a summary of what happens if the matrix A (with real entries) has complex eigendata:

complex eigenvalues (from Friday before break, included here for completeness): Let $A_{2\times 2}$ have complex eigenvalues $\lambda = p \pm q\, i$. For $\lambda = p + q\, i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\, i$. Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$
\begin{aligned}
\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\, i)\\
&= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\, i)\\
&= \left[e^{p\,t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p\,t}\sin(q\,t)\underline{\boldsymbol{b}}\right]\\
&\quad + i\left[e^{p\,t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p\,t}\cos(q\,t)\underline{\boldsymbol{b}}\right] .
\end{aligned}
$$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above. I put the linear combination weights c_1, c_2 on the right instead of the left in the expression below, to facilitate seeing the matrix factorization in the next step:

$$
\begin{aligned}
\underline{\boldsymbol{x}}(t) &= e^{p\,t}[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}]\, c_1\\
&\quad + e^{p\,t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}]\, c_2 .
\end{aligned}
$$

weights

We can rewrite $\underline{\boldsymbol{x}}(t)$ as

$$
\boxed{\underline{\boldsymbol{x}}(t) = e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.}
$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

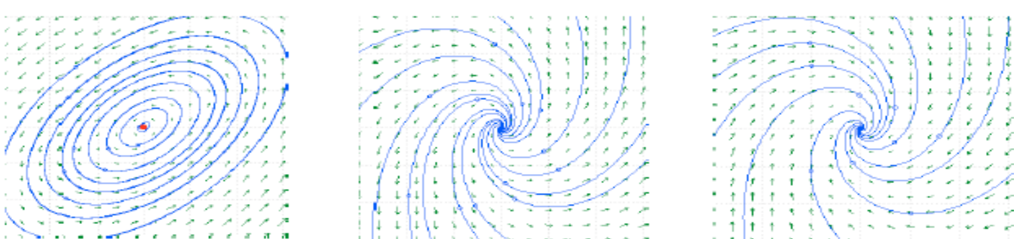
$$
\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

$$
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p\,t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

Thus $\underline{\boldsymbol{x}}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center
Re(λ)=0

spiral source
Re(λ)>0

spiral sink
Re(λ)<0

Real eigenvalues If the matrix $A_{2\times 2}$ is diagonalizable, i.e. if there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of A), then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ is
$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2$$
- And, for each $\underline{\boldsymbol{x}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2$ the value of the tangent field at x is
$$A\,\underline{\boldsymbol{x}} = A\left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2\right) = c_1\lambda_1\underline{\boldsymbol{v}}_1 + c_2\lambda_2\underline{\boldsymbol{v}}_2 .$$

(The text discusses the case of non-diaganalizable A. This can only happen if $det(A - \lambda I) = \left(\lambda - \lambda_1\right)^2$, but the $\lambda = \lambda_1$ eigenspace is defective so that its dimension is one instead of two.)

Exercise 1) Here is an example of what happens when A has two real eigenvalues of opposite sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = -2$
$\lambda_2 = 3$

$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

$A\left(c_1\vec{v}_1 + c_2\vec{v}_2\right) = c_1 A(\vec{v}_1) + c_2 A\vec{v}_2 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$

a) The eigendata for A is

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}, \qquad E_{\lambda=3} = span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

b) Use just the eigendata to sketch the tangent vector field on the plot below. Begin by sketching the two eigenspaces. see next page

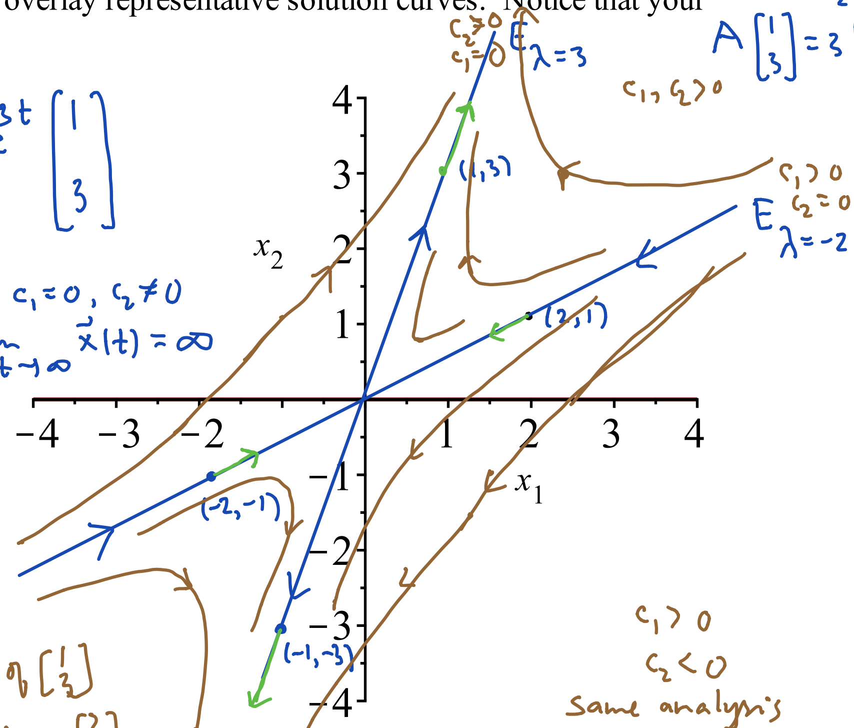
c) Use the general solutions to the DE system to overlay representative solution curves. Notice that your work in b, c is consistent.

$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

if $c_1 \neq 0,\ c_2 = 0$
we're in $E_{\lambda=-2}$
$\lim_{t\to\infty} \vec{x}(t) = 0$

if $c_1 = 0,\ c_2 \neq 0$
$\lim_{t\to\infty} \vec{x}(t) = \infty$



e.g.
$c_1, c_2 > 0$

$\lim_{t\to\infty} \vec{x}(t) = \vec{0} + \vec{\infty}$ in dir of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lim_{t\to-\infty} \vec{x}(t) = \vec{\infty}$ in dir of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

any mult of sol is sol

$c_1 > 0$
$c_2 < 0$
same analysis
then scale & reflect

$$\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

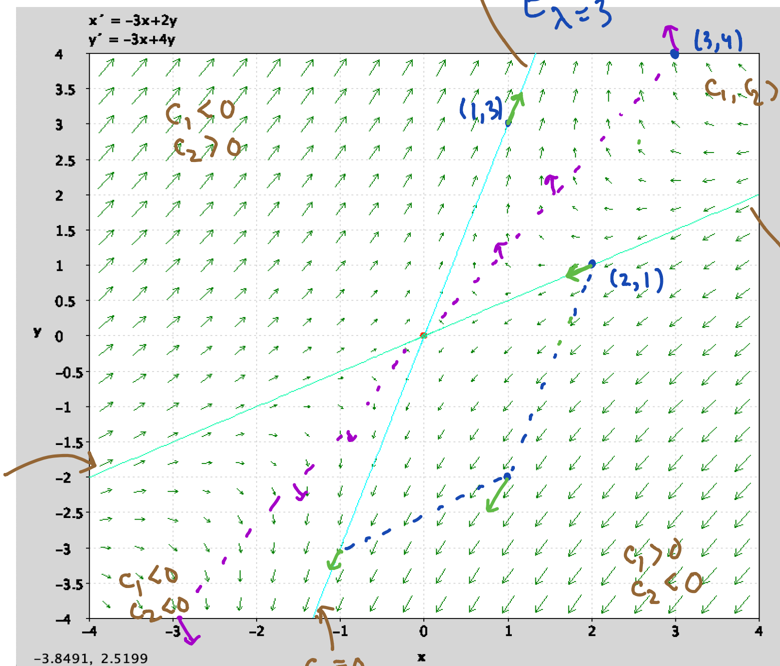
$$A\left(\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix}\right)=A\begin{bmatrix}2\\1\end{bmatrix}+A\begin{bmatrix}1\\3\end{bmatrix}$$

(3,4)

$c_1\vec{v}_1+c_2\vec{v}_2$

$c_2>0$
$c_1=0$

$E_{\lambda=3}$



$E_{\lambda=-2}=\text{span}\{\vec{v}_1\}$

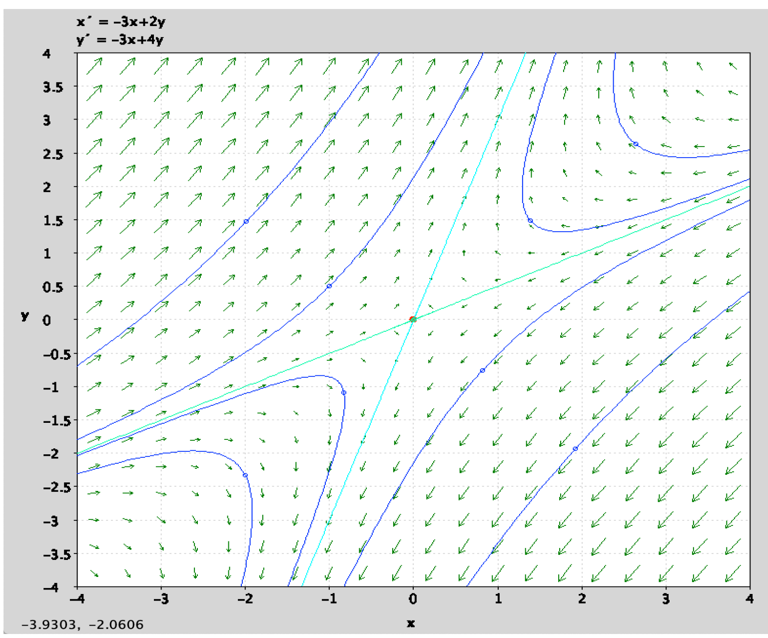
$c_1>0,\ c_2=0$

$c_1<0$
$c_2=0$

$c_1=0$
$c_2<0$

$$A(c_1\vec{v}_1+c_2\vec{v}_2)=c_1\lambda_1\vec{v}_1+c_2\lambda_2\vec{v}_2$$

$$\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$$



Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Use the eigendata of A to find the general solution to the first order system of DE's.

$$E_{\lambda=3} = span\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}, \; E_{\lambda=6} = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves. First sketch the eigenspaces.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

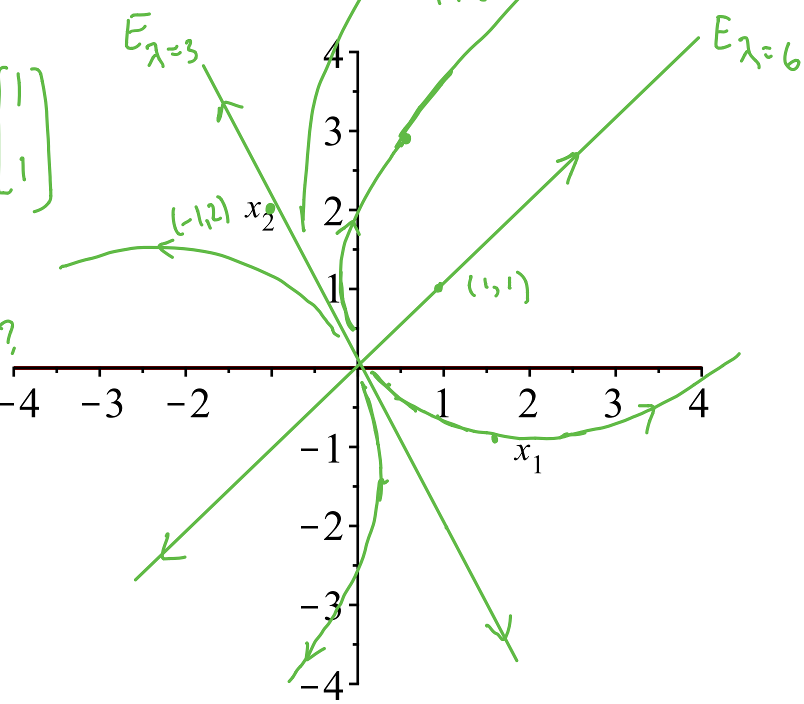
$c_1, c_2 > 0$

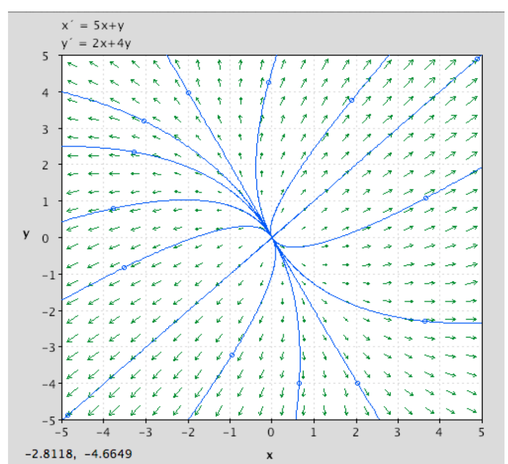
which term dominates $\vec{x}(t)$, $\vec{x}'(t)$ as $t \to \infty$?

$c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

as $t \to -\infty$ which term decays slowest?

$c_1 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$





Theorem: Time reversal: If $\underline{\mathbf{x}}(t)$ solves

$$\underline{\mathbf{x}}' = A\,\underline{\mathbf{x}}$$

then $\underline{\mathbf{z}}(t) := \underline{\mathbf{x}}(-t)$ solves

$$\underline{\mathbf{z}}' = (-A)\underline{\mathbf{z}}$$

proof: by the chain rule,

$$\underline{\mathbf{z}}'(t) = \underline{\mathbf{x}}'(-t)\cdot(-1) = -\underline{\mathbf{x}}'(-t) = -A\,\underline{\mathbf{x}}(-t) = -A\,\underline{\mathbf{z}}.$$

Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A?

b) Describe how the eigendata of the matrix in the system below is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits and solutions are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) If $A\vec{v} = \lambda\vec{v}$

then $(cA)\vec{v} = c\lambda\vec{v}$

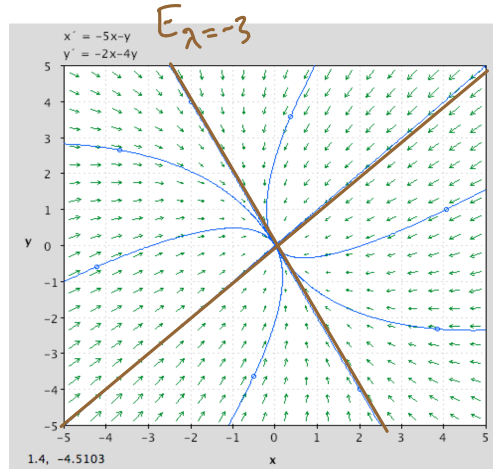
- mult matrix by c
  same eigenvectors, but
  eigenvalue $c\lambda$

b) So if $c=-1$, eigenvalues switch sign,

so the arrows on phase portrait reverse, and the solution curves go in the opposite directions

$E_{\lambda=-3}$

$E_{\lambda=-6}$



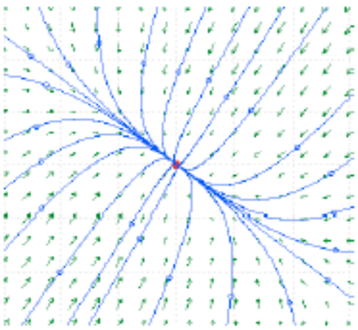
summary: In case the matrix $A_{2\times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\underline{\boldsymbol{x}}'(\underline{t}) = A\,\underline{\boldsymbol{x}}$$

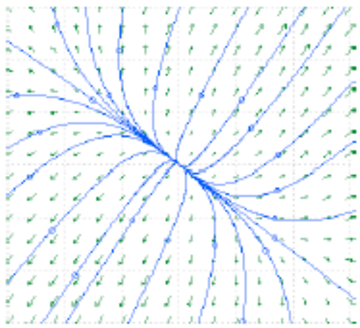
has general solution

$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 \ .$$

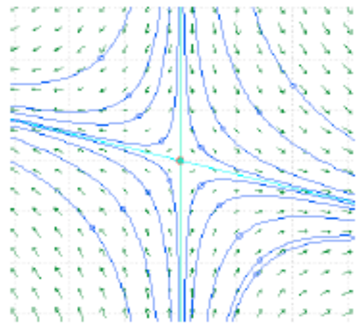
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
$\lambda_1, \lambda_2 < 0$



nodal source
$\lambda_1, \lambda_2 > 0$



saddle point
$\lambda_1 < 0 < \lambda_2$

Could you indentify the eigenspaces, and which ones are $E_{\lambda=\lambda_1}$, $E_{\lambda=\lambda_2}$?