Friday starts here

Definition: Any first order system of differential equations which can be written in the form

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$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

$x'(t) + p(t)x(t) = q(t)$

is called a *first order linear system of DE's*. (Here $\underline{\boldsymbol{x}}(t)$ and $\underline{\boldsymbol{f}}(t)$ are functions from an interval in $\mathbb{R}$, with range lying in $\mathbb{R}^n$, and $P(t)$ is an $n \times n$ matrix whose entries are functions of t. For us $P(t)$ will almost always be a constant matrix. If the system can be written in the form

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

$x'(t) + p(t)x(t) = 0$

we say that the linear system of differential equations is *homogeneous*. Otherwise it is *non-homogeneous* or *inhomogeneous*.

Notice that the operator on vector-valued functions $\underline{\boldsymbol{x}}(t)$ defined by

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{y}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{y}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

$$L(\vec{x}(t) + \vec{y}(t)) = (\vec{x}(t) + \vec{y}(t))' + P(t)[\vec{x}(t) + \vec{y}(t)]$$
$$= \vec{x}' + \vec{y}' + P(t)\vec{x} + P(t)\vec{y}$$
$$= L(\vec{x}(t)) + L(\vec{y}(t))$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is any single particular solution and $\underline{\boldsymbol{x}}_H$ is the general homogeneous solution.

In the case that $P(t) = -A$ is a constant matrix (i.e. entries don't depend on t), we usually write the homogeneous system as

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}.$$

e.g. tank example.

In the case that A is a diagonalizable matrix it turns out we can always find a basis for the homogeneous solution space made of vector-valued functions of the form

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\,\underline{\boldsymbol{v}},$$

where $\underline{\boldsymbol{v}}$ an eigenvector of A and λ is its eigenvalue, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}.$$

System of DE's:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Candidate solution:

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}},$$

where $\underline{\boldsymbol{v}}$ an eigenvector of A and λ is its eigenvalue, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}.$$

We can verify that such an $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ solves the homogeneous DE system above by showing we get a true identity when we substitute it in. We compute the left side of the differential equation:

LHS $\qquad \underline{\boldsymbol{x}}(t) = e^{\lambda t}\vec{v} \quad \Rightarrow \underline{\boldsymbol{x}}'(t) = \lambda\, e^{\lambda t}\underline{\boldsymbol{v}}.$

$$\frac{d}{dt}\begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \vdots \\ e^{\lambda t} v_n \end{bmatrix} = \begin{bmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \\ \vdots \\ \lambda e^{\lambda t} v_n \end{bmatrix} = \lambda e^{\lambda t}\vec{v} \ \checkmark$$

And we compute the right side

RHS $\qquad A\,\underline{\boldsymbol{x}}(t) = A\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}} = e^{\lambda t}\lambda\,\underline{\boldsymbol{v}}.$

Same!

Exercise 3) Use the eigendata of the matrix in our running example solve the initial value problem of Exercise 2!! Compare your solution $\boldsymbol{x}(t)$ to the parametric curve drawn by pplane.

① eigendata for A $(A\vec{v} = \lambda\vec{v} \Leftrightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \Leftrightarrow (A-\lambda I)\vec{v} = \vec{0}$ non-zero sol's iff $A-\lambda I$ is not invertible $|A-\lambda I| = 0)$

$$|A-\lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8 = \lambda^2 + 6\lambda + \cancel{8} - \cancel{8}$$

$$\left((-4-\lambda)(-2-\lambda) - 8 = (-)(-)(\lambda+4)(\lambda+2)\right)$$

$$|A-\lambda I| = \lambda(\lambda+6) = 0$$

roots $\lambda = 0, -6$, are eigenvalues

$E_{\lambda=0} = Nul(A - 0I)$ (or kernel)

eigenspace $\lambda=0$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$1 \cdot col_1 + 2 \cdot col_2 = \vec{0}$$

$$E_{\lambda=0} = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

OR reduce matrix.

$R_1 \to \frac{R_1}{-4}$:
$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$-4R_1 + R_2 \to R_2$
$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

backsolve: $v_1 = \frac{1}{2}v_2$
$v_2 = $ free

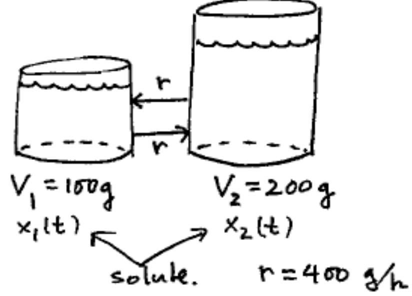
$$\vec{v} = v_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$E_{\lambda=0} = span\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\} = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

$$E_{\lambda=-6} = Nul(A - -6I)$$

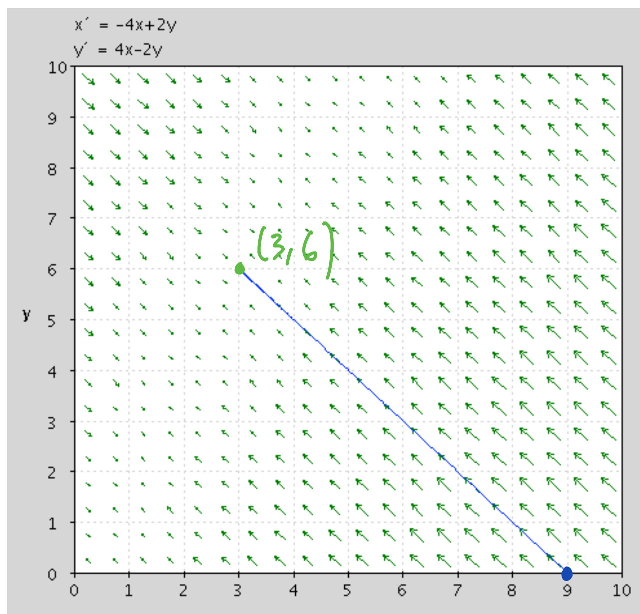
$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{\lambda=-6} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$



$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



sol'tns to $\vec{x}' = A\vec{x}$ $e^{\lambda t}\vec{v}$: $e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

sol'tns
$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

because homog. linear DE

$$\text{IVP}: \vec{x}(0) = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \overset{\text{want}}{=} c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

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5.1-5.2 Systems of differential equations and the vector Calculus we need to study them (5.1). Every differential equation or system of differential equations can be converted into a first order system of differential equations (4.1).

$$\text{soltn } \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Announcements: continue from yesterday...

consistent with graph on previous page

- check: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3+6 \\ 6-6 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$ ✓
- $\lim_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ ✓
- $\vec{x}'(t) = -36e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 36e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

Warm-up Exercise: For the first order system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(from tank problem Wed.)

Show that

$$\vec{z}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{z}_2(t) = e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{are solutions}$$

plug them into the DE system

LHS: $\vec{z}_1'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

RHS $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

LHS = RHS ... $\vec{z}_1(t)$ is a soltn.

LHS: $\vec{z}_2'(t) = -6e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\quad e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(check: $\frac{d}{dt}\begin{bmatrix} -e^{-6t} \\ e^{-6t} \end{bmatrix} = \begin{bmatrix} 6e^{-6t} \\ -6e^{-6t} \end{bmatrix}$

same

RHS: $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-6t}\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= e^{-6t}\begin{bmatrix} 6 \\ -6 \end{bmatrix}$

LHS = RHS

... $\vec{z}_2(t)$ is a soltn.

On Wednesday, we began and maybe finished solving the two-tank IVP example analytically, using eigenvalues and eigenvectors from the matrix A in that problem (!). That is typical of what we will do in section 5.2, to solve the first order system of DE's

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= A\,\underline{\mathbf{x}} \\ \underline{\mathbf{x}}(0) &= \underline{\mathbf{x}}_0 \,. \end{aligned}$$

At this point though, it's a good idea to review and extend some differentiation rules you probably learned in multivariable Calculus, when you studied the calculus of parametric curves. This is related to material in section 5.1 of the text, e.g. page 271. Most of the rest of section 5.1 is material you learned in Math 2270 - you may wish to scan it to make sure it's still familiar.

1) If $\underline{\mathbf{x}}(t) = \underline{\boldsymbol{b}}$ is a constant vector, then $\underline{\mathbf{x}}'(t) = \underline{\mathbf{0}}$ for all t, and vise-verse. (Because all of the entries in the vector $\underline{\boldsymbol{b}}$ are constants, and their derivatives are zero. And if the derivatives of all entries of a vector are identically zero, then the entries are constants.)

2) Sum rule for differentiation:

$$\frac{d}{dt}\left(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)\right) = \underline{\mathbf{x}}'(t) + \underline{\mathbf{y}}'(t): \qquad \text{Both sides simplify to} \begin{bmatrix} x_1'(t) + y_1'(t) \\ x_2'(t) + y_2'(t) \\ \vdots \\ x_n'(t) + y_n'(t) \end{bmatrix}$$

3) Constant multiple rule for differentiation:

$$\frac{d}{dt}\left(c\,\underline{\mathbf{x}}(t)\right) = c\,\underline{\mathbf{x}}'(t): \qquad \text{Both sides simplify to} \quad c \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

4) Matrix-valued functions sometimes show up and sometimes need to be differentiated. This is done with the limit definition, and amounts to differentiating each entry of the matrix. For example, if $A(t)$ is a 2×2 matrix, then

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} a_{11}(t+\Delta t) & a_{12}(t+\Delta t) \\ a_{21}(t+\Delta t) & a_{22}(t+\Delta t) \end{bmatrix} - \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\begin{bmatrix} a_{11}(t+\Delta t)-a_{11}(t) & a_{12}(t+\Delta t)-a_{12}(t) \\ a_{21}(t+\Delta t)-a_{21}(t) & a_{22}(t+\Delta t)-a_{22}(t) \end{bmatrix}$$

$$= \lim_{\Delta t \to 0}\begin{bmatrix} \dfrac{a_{11}(t+\Delta t)-a_{11}(t)}{\Delta t} & \dfrac{a_{12}(t+\Delta t)-a_{12}(t)}{\Delta t} \\ \dfrac{a_{21}(t+\Delta t)-a_{21}(t)}{\Delta t} & \dfrac{a_{22}(t+\Delta t)-a_{22}(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}.$$

5) The constant rule (1), sum rule (2), and constant multiple rule (3) also hold for matrix derivatives.

we'll discuss this more carefully on Monday...

Universal product rule: Shortcut to take the derivatives of

$f(t)\underline{\mathbf{x}}(t)$ (scalar function times vector function),

$f(t)A(t)$ (scalar function times matrix function),

$A(t)\underline{\mathbf{x}}(t)$ (matrix function times vector function),

$\underline{\mathbf{x}}(t) \cdot \underline{\mathbf{y}}(t)$ (vector function dot product with vector function),

$\underline{\mathbf{x}}(t) \times \underline{\mathbf{y}}(t)$ (cross product of two vector functions),

$A(t)B(t)$ (matrix function times matrix function).

As long as the "product" operation distributes over addition, and scalars times the product equal the products where the scalar is paired with either one of the terms, there is a product rule. Since the product operation is not assumed to be commutative you need to be careful about the order in which you write down the terms in the product rule, though.

Theorem. Let $A(t)$, $B(t)$ be differentiable scalar, matrix or vector-valued functions of t, and let $*$ be a product operation as above. Then

$$\frac{d}{dt}(A(t)*B(t)) = A'(t)*B(t) + A(t)*B'(t).$$

The explanation just rewrites the limit definition explanation for the scalar function product rule that you learned in Calculus, and assumes the product distributes over sums and that scalars can pass through the product to either one of the terms, as is true for all the examples above. It also uses the fact that differentiable functions are continuous, that you learned in Calculus. Here is one explanation that proves all of those product rules at once:

$$\frac{d}{dt}(A(t)*B(t)) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t) + A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t+\Delta t) - A(t+\Delta t)*B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t)*B(t) - A(t)*B(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(A(t+\Delta t)*\left(B(t+\Delta t) - B(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)*B(t)$$

$$= \lim_{\Delta t \to 0}\left(A(t+\Delta t)*\left(\frac{1}{\Delta t}\left(B(t+\Delta t) - B(t)\right)\right)\right) + \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\left(A(t+\Delta t) - A(t)\right)\right)*B(t)$$

$$= A(t)*B'(t) + A'(t)*B(t).$$