As a concrete prototype for how mathematical modeling works, consider:

to give credibility.

Exercise 3) *Newton's law of cooling* is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ is assumed to change at a rate proportional to ~~to~~ the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use the assumptions in the model above, to "derive" (i.e. explain) the differential equation for the $T(t)$ of the object being heated or cooled:

$$\frac{dT}{dt} = -k(T - A)\,.$$

we used $-k$ so that we could take $k>0$

b) Would the model have been correct if we wrote $\dfrac{dT}{dt} = k(T - A)$ instead?

would be fine, except k would be <0 so that $T'(t)<0$ when $A<T$, $T'(t)>0$ when $A>T$

c) Use the Newton's law of cooling model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 $^\circ$F. An hour later the body temperature has decreased to 60°. It's been a winter inversion in SLC, with constant ambient temperature 30°. Assuming the Newton's law model, estimate the time of death. Hint: Begin by finding formulas for the functions $T(t)$ that solve this "separable" differential equation.

1) Solve $\dfrac{dT}{dt} = -k(T-A)$

separable $\displaystyle\int \frac{dT}{T-A} = \int -k\,dt$

$$\ln|T-A| + C_1 = -kt + C_2$$

$$e^{\ln|T-A|} = e^{-kt + C_3}$$

$$|T-A| = e^{-kt}\underbrace{e^{C_3}}_{>0}$$

$$T - A = \underbrace{\pm e^{C_3}}_{C} e^{-kt}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

3 unknowns A, C, k

$\|$ 30°

3 pieces of data to determine them

...

Wed

$t=0$ be 3:00 pm. measure time in hours

$$T(t) = 30 + Ce^{-kt}$$

$$T(0) = 70 = 30 + C$$

$$T(1) = 60 = 30 + Ce^{-k}$$

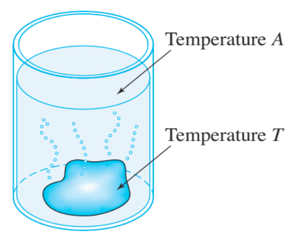


FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

So $C = 40$.

$$T(1) = 60 = 30 + 40e^{-k}$$
$$30 = 40e^{-k}$$
$$.75 = e^{-k}$$

$\left(\ln(.75) = -k \quad k = -\ln(.75) \right)$

So
$$T(t) = 30 + 40\left(e^{-k}\right)^t$$
$$\boxed{T(t) = 30 + 40(.75)^t}$$

Set $T = 98.6$, solve for t

$$98.6 = 30 + 40(.75)^t$$
$$\frac{68.6}{40} = \cancel{40}(.75)^t$$
$$\ln\left(\frac{68.6}{40}\right) = t \cancel{\ln(.75)}$$
$$\boxed{\frac{\ln\left(\frac{68.6}{40}\right)}{\ln(.75)}}$$
$$-1.875 = t$$

clock time $3 - 1.875 = 1.125$ O'clock

$\approx 1{:}075$ O'clock.

Section 1.4 is about *separable* differential equations which is a generalization that includes those of section 1.2:

Definition: A *separable* first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

Solution (chain-rule justified): One can rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0) .$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Taking antiderivatives with respect to the variable x we have

$$\int g(y)\frac{dy}{dx}\, dx = \int f(x)\, dx.$$

If $G(y)$ is any antiderivative of $g(y)$ with respect to the variable y then

$$g(y(x))\frac{dy}{dx} = G'(y(x))y'(x)$$

which by the chain rule (read backwards) is precisely

$$\frac{d}{dx}G(y(x)).$$

So we have

$$\int \frac{d}{dx}G(y(x))dx = \int f(x)\, dx ,$$

which we antidifferentiate with respect to x and obtain

$$G(y(x)) = F(x) + C.$$

where $F(x)$ is any particular antiderivative of $f(x)$. This identity

$$G(y) = F(x) + C$$

expresses solutions $y(x)$ *implicitly* as functions of x . (By differentiating this identity implicitly as you did in Calculus, you recover the original differential equation.)

You may be able to use algebra to solve this equation *explicitly* for $y = y(x)$ as we did for $T = T(t)$ in the Newton's Law of cooling problem.

<u>Solution (differential magic for doing the computation quickly)</u>: Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx \,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx \text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C \,. \quad \text{Agrees!}$$

This differential magic is related to the "method of substitution" in antidifferentiation, which is essentially the "chain rule in reverse" for integration techniques.

Discuss on Wed.

Exercise 2: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

a) Use separation of variables to find solutions to this DE.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a?
c) Explain why the IVP

$$\frac{dy}{dx} = 1 + y^2$$
$$y(0) = 0$$

has a solution, but this solution does not exist for all x.

Wed warmup:

$$y(x) = \tan x = \frac{\sin x}{\cos x}$$

$\cos x = 0$ at $\pm \frac{\pi}{2}, \pm \frac{3}{2}\pi, \ldots$

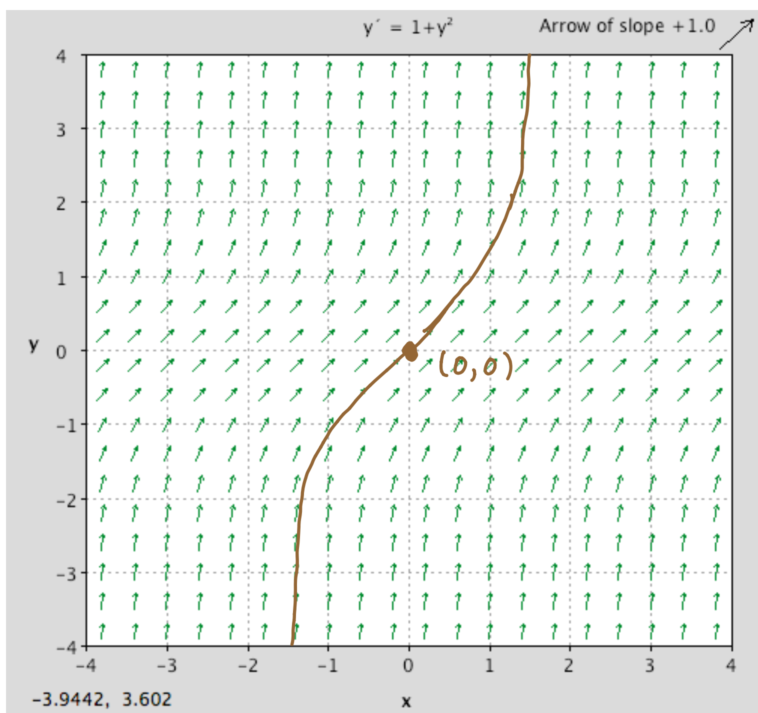
You can download the java applet "dfield" from the Rice University URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

(c) largest interval containing $x_0 = 0$ soln to IVP exists on

$-\pi/2 < x < \pi/2$

slope fcn at $(x, y) = 1 + y^2$

(b)



Section 1.2 applications:

An important class of antidifferentiation differential equations applications arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

$m x''(t) = \text{net forces}$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

$$v(t) = x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad \frac{m}{sec} \qquad a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} \quad \frac{m/s}{s}$$

$$= x''(t)$$

Exercise 3:

a) If the units for position are meters m and the units for time are seconds s , what are the units for velocity and acceleration? (These are *mks* units.)

b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

velocity: $\frac{m}{sec}$ or $\frac{ft}{sec}$.   accel $\frac{m}{s^2}$ $\frac{ft}{s^2}$

Exercise 4: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof. (Use the approximate value for the acceleration due to gravity, $g = 9.8 \ \frac{m}{s^2}$.)

a) Neglecting friction, how high will the object get above ground?

b) When does the object land ?

$y(5.102) \approx 157.6$ m

ans 10.2 sec (8-6.12)

Exercise 5:
Suppose the acceleration function is a negative constant $-a$,
$$x''(t) = -a\,.$$

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.
b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is
$$x_{max} = \frac{1}{2}\frac{v_0^2}{a}\,.$$
(This formula may help with some homework problems.)

this is in HW.

c) Adapt the answer to b to check part of your work in Exercise 4.

5a) $x''(t) = -a \qquad (a>0)$

$$x'(t) = \int x''(t)\,dt = \int -a\,dt = -at + C.$$

$$v(t) = x'(t) = -at + C$$

@ $t=0$ $\quad v(0) = 0 + C \Rightarrow C = v(0) = v_0$

$$\boxed{x'(t) = -at + v_0}$$

So $\quad x(t) = \int -at + v_0\,dt = -\frac{a}{2}t^2 + v_0 t + C$

@ $t=0$: $\quad x(0) = x_0 = 0 + 0 + C$

$$\boxed{x(t) = -\frac{a}{2}t^2 + v_0 t + x_0}$$

Math 2280-002
Wed Jan 9
1.3-1.4 more slope fields; existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Announcements:
- For today's quiz, you'll solve a separable DE.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

...

- Finish M, T notes today
  on Fri, we'll do Wed notes

'til 12:57

Warm-up Exercise: Use separation of variables to solve the initial value problem for y(x):

$$\begin{cases} \frac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$$

ans $\boxed{y = \tan x}$!

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\arctan(y) = x + C$$

tan: $y = \tan(x+C)$

@ $x=0$ $\quad 0 = \tan(C)$

can choose $C = 0$.
since $\tan(0) = 0$

$$\left(\begin{aligned} & \sin^2 x + \cos^2 x = 1 \\ \div \cos^2 x: \quad & \tan^2 x + 1 = \sec^2 x \\ & (y^2 + 1 = y') \end{aligned} \right)$$