2e) How did someone find formulas for the solution functions in part 2a? Or how could we have found them if they weren't given to us?

$$\boxed{\frac{dy}{dx} = y^2}$$

Tuesday start here

Answer: They used the chain rule in reverse .... the systematic way of doing this is called "separation of variables", section 1.4, which we'll discuss in more detail tomorrow and which many of you discussed in a prerequisite Calculus class. Let's work the "chain rule in reverse" for this example in order to recall ...

1st try might be to antidifferentiate with respect to x

$$\frac{dy}{dx} = y(x)^2$$

don't know y(x) so can't antidiff RHS (right hand side)

2nd try: $\frac{y'(x)}{y(x)^2} = 1 \qquad (y(x) \neq 0)$

now $\int \frac{y'(x)}{y(x)^2}\,dx = \int 1\,dx$

let $u(x) = y(x)$

$du = y'(x)\,dx$

$\int \frac{du}{u^2}$

$= -\frac{1}{u} + C$

$$\boxed{= -\frac{1}{y(x)} + C = x + D}$$

$= -\frac{1}{y(x)} = x + E \qquad (E = D - C)$

So $y(x) = -\frac{1}{x+E} = \frac{1}{-x-E}$

we were told

$y(x) = \frac{1}{C-x}$

equivalent formulas

differentials shortcut (algorithm for separation of variables)

$$\frac{dy}{dx} = y^2$$

$$dy = y^2\,dx$$

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int 1 \cdot dx$$

$$\boxed{-\frac{1}{y} + C = x + D}$$

• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example, you've seen the *exponential growth/decay differential equation*, modeling situations in which *The rate of change of the quantity $P(t)$ is proportional to $P(t)$*:

$$P'(t) = k\,P(t)$$

with solutions

$$P(t) = P_0\,e^{k\,t}.$$

And you've seen the *constant acceleration* particle motion differential equation

$$y''(t) = a \quad \text{(constant)}$$

with solutions

$$y(t) = y_0 + v_0\,t + \frac{a}{2}t^2.$$

We'll see many more differential equations applications in this class. The general modeling paradigm and feedback loop is discussed in our text in section 1.1:

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

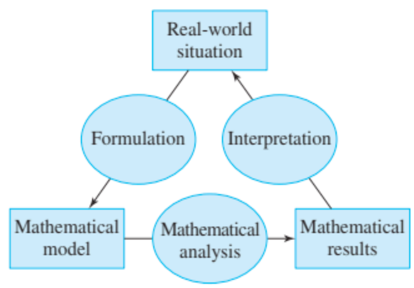

FIGURE 1.1.4. The process of mathematical modeling.

For example, the exponential growth model is effective for continuous compound interest in bank accounts, bacterial growth with no resource constraints, and radioactive decay (negative k).   And the constant acceleration model is effective when there are no drag forces on the object and the underlying acceleration is close to a constant. But more sophisticated models are needed if the situation is less simple, e.g. the *logisitic* population model for populations with resource constraints, and particle acceleration models that need to take into account drag forces and non-constant background acceleration force. We will study such modifications in Chapter 2.

As a concrete prototype for how mathematical modeling works, consider:

to give credibility.

Exercise 3) *Newton's law of cooling* is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ is assumed to change at a rate proportional to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use the assumptions in the model above, to "derive" (i.e. explain) the differential equation for the $T(t)$ of the object being heated or cooled:

$$\frac{dT}{dt} = -k(T - A).$$

we used $-k$ so that we could take $k > 0$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

would be fine, except k would be < 0 so that $T'(t) < 0$ when $A < T$, $T'(t) > 0$ when $A > T$

c) Use the Newton's law of cooling model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is $70\,^\circ$F. An hour later the body temperature has decreased to $60\,^\circ$. It's been a winter inversion in SLC, with constant ambient temperature $30\,^\circ$. Assuming the Newton's law model, estimate the time of death. Hint: Begin by finding formulas for the functions $T(t)$ that solve this "separable" differential equation.

1) Solve $\frac{dT}{dt} = -k(T - A)$

separable $\int \frac{dT}{T-A} = \int -k\, dt$

$$\ln|T-A| + C_1 = -kt + C_2$$

$$e^{\ln|T-A|} = e^{-kt + C_3}$$

$$|T-A| = e^{-kt}\underbrace{e^{C_3}}_{>0}$$

$$T - A = \underbrace{\pm e^{C_3}}_{C} e^{-kt}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

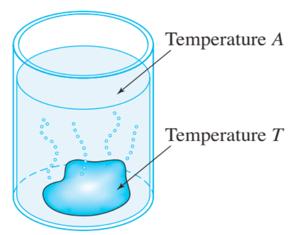

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

3 unknowns A, C, k ($A = 30^\circ$)

3 pieces of data to determine them

...

Math 2280-1
Tues Jan 8
1.2, 1.4 antidifferentiation differential equations and separable differential equations

Announcements: HW you can try after what we discuss today includes

1.1 6, 15, 27, 29, 30, 34
1.2 w1.1 abc
2, 6, 7, 9
1.3 2, 6
w1.3
1.4 4, 12, 20

• Quiz tomorrow

I'll usually start class with a warm-up exercise (starting 5 minutes early)

'til 12:57

Warm-up Exercise: Use antidifferentiation to solve the initial value problem for $y(x)$

$$\text{IVP} \begin{cases} \dfrac{dy}{dx} = x\sqrt{x^2+4} \\ y(0) = 0 \end{cases}$$

soln $y(x) = \int x\sqrt{x^2+4}\,dx$ $= \frac{1}{3}(x^2+4)^{3/2} - \frac{8}{3}$ $+C$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$y(x) = \int u^{1/2}\,\tfrac{1}{2}du$$
$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C$$
$$y = \frac{1}{3}(x^2+4)^{3/2} + C$$
$$y(0) = 0 = \frac{1}{3}4^{3/2} + C$$
$$-\frac{1}{3}4^{3/2} = C$$
$$-8/3 = C$$

check! $y(0) = \frac{1}{3}4^{3/2} - \frac{8}{3} = \frac{1}{3}8 - \frac{8}{3} = 0$ ✓

$y'(x) = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(x^2+4)^{1/2}\cancel{2}x$

$= x\sqrt{x^2+4}$ ✓

Section 1.2 is about differential equations equivalent to ones of the form

$$\frac{dy}{dx} = f(x)$$

which we solve by direct antidifferentiation, as you learned in Calculus.

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$
$$y(0) = 0$$

this was our warmup

Section 1.4 is about *separable* differential equations which is a generalization that includes those of section 1.2:

Definition: A *separable* first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

Solution (chain-rule justified): One can rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0\text{)} .$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Taking antiderivatives with respect to the variable x we have

$$\int g(y)\frac{dy}{dx}\, dx = \int f(x)\, dx.$$

If $G(y)$ is any antiderivative of $g(y)$ with respect to the variable y then

$$g(y(x))\frac{dy}{dx} = G'(y(x))y'(x)$$

which by the chain rule (read backwards) is precisely

$$\frac{d}{dx}G(y(x)).$$

So we have

$$\int \frac{d}{dx}G(y(x))dx = \int f(x)\, dx ,$$

which we antidifferentiate with respect to x and obtain

$$G(y(x)) = F(x) + C.$$

where $F(x)$ is any particular antiderivative of $f(x)$. This identity

$$G(y) = F(x) + C$$

expresses solutions $y(x)$ *implicitly* as functions of x . (By differentiating this identity implicitly as you did in Calculus, you recover the original differential equation.)

You may be able to use algebra to solve this equation *explicitly* for $y = y(x)$ as we did for $T = T(t)$ in the Newton's Law of cooling problem.

Solution (differential magic for doing the computation quickly): Treat $\frac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx \,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx \text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C \,. \quad \text{Agrees!}$$

This differential magic is related to the "method of substitution" in antidifferentiation, which is essentially the "chain rule in reverse" for integration techniques.