Math 2280-002 Week 4 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes include material from 2.3-2.6, with an introduction to 3.1-3.2.

Mon Jan 28
2.3 Improved acceleration models: linear and quadratic drag forces.

Announcements: • I added 1.5.38 to your HW on Friday.

Friday solute amt $x(t)$

$x'(t) = r_i c_i - r_o c_o$ If $r_i = r_o$

$x'(t) = r c_i - \frac{r}{V} x$

alpha rho

α, ρ are constants

'til 12:57

Warm-up Exercise: Solve for velocity $v(t)$

$$\begin{cases} v'(t) = \alpha - \rho v \\ v(0) = v_0 \end{cases}$$

• This is your friend, the constant coef. 1st order linear DE. You want to be able to solve it blindfolded.

solution

$$v(t) = \frac{\alpha}{\rho} + \left(v_0 - \frac{\alpha}{\rho}\right) e^{-\rho t}$$

Check:

$v' + \rho v = \alpha$ In linear DE form

(2270) Solns: $v = v_p + v_H$

particular

solves $v' + \rho v = 0$ homog. soln.

$$v = \frac{\alpha}{\rho} + \underbrace{\left(v_0 - \frac{\alpha}{\rho}\right)}_{\text{"C"}} e^{-\rho t}$$

particular soln

homog. solns

$$v' + \rho v = \alpha$$
$$e^{\rho t}(v' + \rho v) = \alpha e^{\rho t}$$
$$\frac{d}{dt}\left(e^{\rho t} v\right) = \alpha e^{\rho t}$$
$$\int: \quad e^{\rho t} v = \int \alpha e^{\rho t} dt$$
$$e^{\rho t} v = \frac{\alpha}{\rho} e^{\rho t} + C$$
$$\div e^{\rho t} \quad v = \frac{\alpha}{\rho} + \underbrace{C}_{v_0 - \frac{\alpha}{\rho}} e^{-\rho t}$$

$$v(0) = v_0 = \frac{\alpha}{\rho} + C \Rightarrow C = v_0 - \frac{\alpha}{\rho}$$

$$\boxed{v(t) = \frac{\alpha}{\rho} + \left(v_0 - \frac{\alpha}{\rho}\right) e^{-\rho t}}$$

2.3 Improved acceleration models: velocity-dependent drag

For 1-dimensional particle motion, with

$$\text{position } x(t) \text{ (or } y(t)\text{)},$$
$$\text{velocity } x'(t) = v(t)\text{, and}$$
$$\text{acceleration } x''(t) = v'(t) = a(t)$$

We have Newton's 2^{nd} law

$$m\,v'(t) = F$$

where F is the net force, possibly a sum consisting of several terms.

- By now we're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\,t + v_0$$
$$x(t) = \frac{1}{2}\alpha\,t^2 + v_0\,t + x_0\,.$$

Examples we've seen already in this course

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- ~~α arising from a charged particle moving through a constant electric field (lab problem)~~
- boats or cars moving with constant acceleration or deceleration (homework).

New today !!! Combine a constant background force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\,v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\,|v|^p\ ,\ 1 \le p \le 2\,.$$

$F_R.$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\,v'(t) = m\,\alpha - k\,v$$

This linear model makes sense for "slow" velocities, as a <u>linearization</u> of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

neglect if v small enough.

$$F_R(v) = \underbrace{\cancel{F_R(0)}}_{=0} + F_R'(0)\,v + \cancel{\frac{1}{2!}F_R''(0)v^2 + \ldots}$$

$F_R(0) = 0$ and <u>for small enough v the higher order terms might be negligable compared to the linear term</u>, so

$$F_R(v) \approx \underbrace{F_R'(0)}\,v \approx \underbrace{-k}\,v\ .$$

We write $-k\,v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

<u>Exercise 1</u>: Let's rewrite the linear drag model

$$m\,v'(t) = m\,\alpha - k\,v$$

as

$$v'(t) = \alpha - \rho\,v$$

• see warmup.

where the $\rho = \dfrac{k}{m}$. Now construct the phase diagram for v . (Hint: there is one critical value for v.)

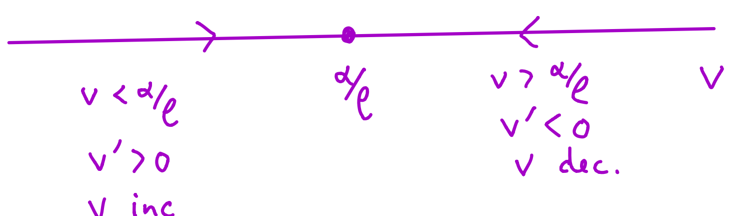
equil. sol'tns, arrows for inc/dec.

roots of "slope" fcn $\quad \alpha - \rho v = 0$

$\alpha = \rho v$

$v = \dfrac{\alpha}{\rho}$

$v' = \alpha - \rho v$
$= -\rho\,(v - \alpha/\rho)$



So. $\lim\limits_{t\to\infty} v(t) = \dfrac{\alpha}{\rho}$ no matter value of v_0

The value of the constant velocity solution is called the *terminal velocity,* which makes good sense when you think about the underlying physics and phase diagram.

$\alpha/\rho = v_T$ terminal veloc.

think of falling objects (see HW)

• $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number" Accounting for the fact that the resistance opposes direction of motion we get

Resistance force is opposite sign of vel.

$$m\, v'(t) = m\,\alpha - k\, v^2 \quad \text{if } v > 0$$

$$m\, v'(t) = m\,\alpha + k\, v^2 \quad \text{if } v < 0.$$

Do you understandt the sign of the drag terms in these two cases?

http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity

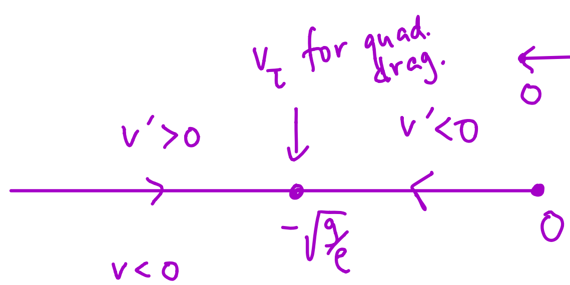
Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

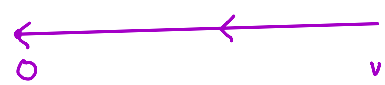
$$\left.\begin{array}{ll} v'(t) = \alpha - \rho\, v^2 & \text{if } v > 0 \\ v'(t) = \alpha + \rho\, v^2 & \text{if } v < 0. \end{array}\right\}$$

Exercise 2) Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction.

$$\left.\begin{array}{ll} v'(t) = -g - \rho\, v^2 & \text{if } v > 0 \\ v'(t) = -g + \rho\, v^2 & \text{if } v < 0. \end{array}\right\}$$

Draw the phase diagrams. Note that each diagram contains a half line of v-values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \le 0$. Is there a terminal velocity?





$v' = -g - \rho v^2$

$v > 0 \Rightarrow v' < 0$

v dec.

$v < 0$

$v' = -g + \rho v^2 = \rho\left(v^2 - \frac{g}{\rho}\right)$

equil sol: $-g + \rho v^2 = 0$, $v^2 = \frac{g}{\rho}$, $v = -\sqrt{\frac{g}{\rho}}$

How would you set up and get started on finding the solutions to these two differential equations? A couple of your homework problems are related to this quadratic drag model.

$\frac{dv}{dt} = -g - \rho v^2 = -g\left(1 + \frac{\rho}{g} v^2\right)$ separable $v(t) > 0$

$\int \frac{dv}{1 + \frac{\rho}{g} v^2} = \int -g\, dt$

$u = \sqrt{\frac{\rho}{g}}\, v$ $\quad du = \sqrt{\frac{\rho}{g}}\, dv$ $\quad \sqrt{\frac{g}{\rho}}\, du = dv$

$\sqrt{\frac{g}{\rho}} \int \frac{1}{1 + u^2}\, du$ use arctan.

$\frac{dv}{dt} = -g + \rho v^2 = \rho\left(v^2 - \frac{g}{\rho}\right)$ $v(t) < 0$

$\frac{dv}{v^2 - \frac{g}{\rho}} = \rho\, dt$

$\left(v - \sqrt{g/\rho}\right)\left(v + \sqrt{g/\rho}\right)$ part fracs

Exercise 3a Returning to the linear drag model and with. Solve the IVP

$$v'(t) = \alpha - \rho v = -\rho\left(v - \frac{\alpha}{\rho}\right)$$
$$v(0) = v_0$$

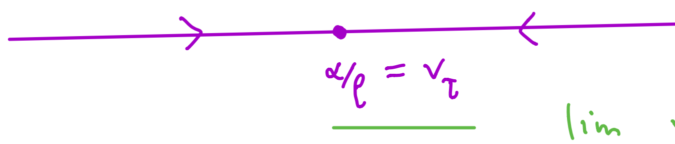
and verify that your solutions are consistent with the phase diagram analysis two pages back. (This is, once again, our friend the first order constant coefficient linear differential equation.)

warm-up!

decaying exponential

$$v(t) = \frac{\alpha}{\rho} + \left(v_0 - \frac{\alpha}{\rho}\right)e^{-\rho t}$$

consistent



$$\lim_{t\to\infty} v(t) = v_\tau$$

3b integrate the velocity function above to find a formula for the position function $y(t)$. Write $y(0) = y_0$.

$$y(t) = \int v(t)\,dt$$
$$= \int \frac{\alpha}{\rho} + \left(v_0 - \frac{\alpha}{\rho}\right)e^{-\rho t}\,dt$$

$$y(t) = \frac{\alpha}{\rho}t + \left(v_0 - \frac{\alpha}{\rho}\right)\frac{e^{-\rho t}}{-\rho} + C$$

$$y(t) = v_\tau t + \frac{(v_0 - v_\tau)}{-\rho}e^{-\rho t} + C$$

$$@\, t = 0 : \quad y_0 = 0 - \frac{(v_0 - v_\tau)}{\rho} + C$$

$$C = y_0 + \left(\frac{v_0 - v_\tau}{\rho}\right)$$

$$y(t) = y_0 + v_\tau t + \left(\frac{v_0 - v_\tau}{\rho}\right)\left(1 - e^{-\rho t}\right)$$

Comparison of Calc 1 constant acceleration vs. linear drag acceleration model:

We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial velocity of 49 $\frac{m}{s}$. (That initial velocity is chosen because its numerical value is 5 times the numerical value of $g = 9.8\ \frac{m}{s^2}$, which simplifies some of the computations.) In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of coupled differential equations for the horizontal and vertical motions, if you didn't shoot the bolt straight up. So we're shooting it straight up.

No drag:

$$v'(t) = -g \approx -9.8\ \frac{m}{s^2}$$

$$v(t) = -g\,t + v_0 = -g\,t + 5\,g = g\cdot(-t+5) \quad \frac{m}{s}$$

$$x(t) = -\frac{1}{2}g\,t^2 + v_0\,t + x_0 = -\frac{1}{2}g\,t^2 + 5\,g\,t = g\,t\left(-\frac{1}{2}t + 5\right) \quad m$$

So our deadbolt goes up for 5 seconds, then drops for 5 seconds until it hits the ground. Its maximum height is given by

$$x(5) = \frac{g\cdot 5\cdot 5}{2} = 122.5\ m$$

Linear drag: The same deadbolt, with the same initial velocity with numerical value $5\,g = 49\ \frac{m}{s}$. We're told that our deadbolt has a measured terminal velocity of $v_\tau = -245\ \frac{m}{s}$ which is the numerical value of $-25\,g$. The initial value problem for velocity is

$$v'(t) = -g - \rho\,v$$
$$v(0) = v_0 = 25\,g = 245.$$

So, in these letters the terminal velocity is (easily recoverable by setting $v'(t) = 0$) and is given by

$$v_\tau = -\frac{g}{\rho} = -25\,g \Rightarrow \rho = .04\ .$$

So, from our earlier work: Substituting $\alpha = -g$ into the formulas for terminal velocity, velocity, and height:

$$v(t) = v_\tau + \left(v_0 - v_\tau\right)\mathrm{e}^{-\rho\,t} = -245 + 294\ \mathrm{e}^{-.04\,t}\ .$$

$$y(t) = -245\,t + \frac{294}{.04}\left(1 - \mathrm{e}^{-.04\,t}\right).$$

The maximum height occurs when $v(t) = 0$,

$$-245 + 294\, e^{-.04\, t} = 0$$

which yields $t = 4.56$ sec:

> $-\dfrac{\ln\left(\dfrac{245.}{294.}\right)}{.04}$;

$$4.558038920 \quad \textbf{(1)}$$

And the maximum height is 108.3 m:

> $y := t \to -245.\cdot t + \dfrac{294}{.04}\left(1 - e^{-.04\cdot t}\right)$;
$y(4.558038920)$;

$$y := t \to (-1)\cdot 245.\, t + \frac{294\left(1 - e^{(-1)\cdot 0.04\, t}\right)}{0.04}$$

$$108.280465 \quad \textbf{(2)}$$

So the drag caused the deadbolt to stop going up sooner ($t = 4.56$ vs. $t = 5$ sec) and to not get as high (108.3 *vs* 122.5 *m*). This makes sense. It's also interesting what happens on the way down - the drag makes the descent longer than the ascent: 4.85 seconds on the descent, vs. 4.56 on the ascent.

> $solve(y(t) = 0, t)$;

$$9.410949931,\ 0. \quad \textbf{(3)}$$

> $9.411 - 4.558$;

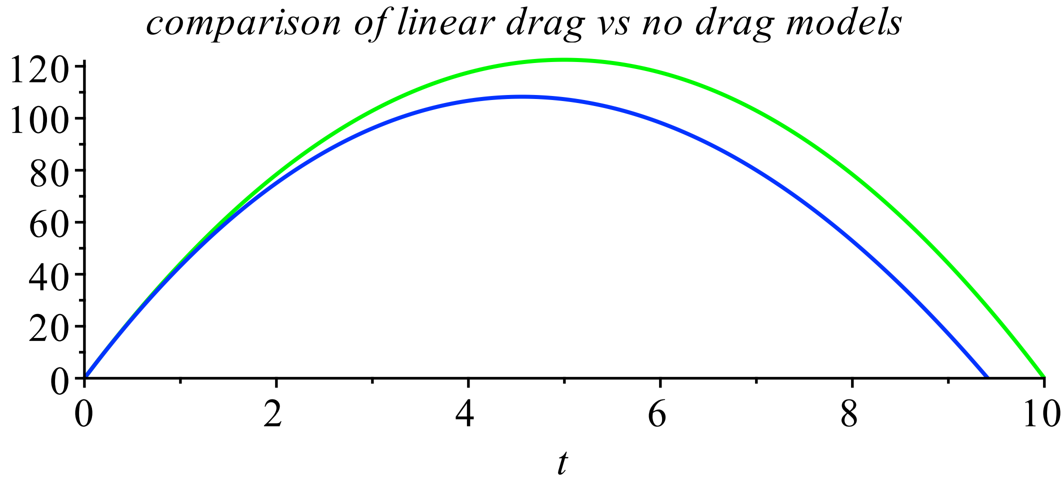
$$4.853 \quad \textbf{(4)}$$

IMPORTANT to note that we needed to use a "solve" command (or something sophisticated like Newton's method) to find when the deadbolt landed. You cannot isolate the t algebraically when trying to solve $y(t) = 0$ for t. This situation will also happen in some of the lab and/or homework problems this week.

$$-245.\cdot t + \frac{294}{.04}\left(1 - e^{-.04\cdot t}\right) = 0$$

picture:

> $z := t \rightarrow 49\,t - 4.9 \cdot t^2$:
> $with(plots)$:
> $plot1 := plot(z(t), t = 0..10, color = green)$:
> $plot2 := plot(y(t), t = 0..9.4110, color = blue)$:
> $display(\{plot1, plot2\}, title = \text{`comparison of linear drag vs no drag models`});$



>