Wed warmup (in Tuesday notes): Read over this application exercise. We'll use previous work for solution to constant coeff IVP for $x(t)$:

$$\text{IVP} \begin{cases} x' + ax = b \\ x(0) = x_0 \end{cases} \longrightarrow \text{solution } x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}$$

Exercise 3 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3, and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?

background concentration c

at $t = 0$, in Erie, concentration $= 5c$

$V = 480\ km^3$ const

$r_i = r_o = r = 350\ km^3/\text{year}$

$x(t)$ = poll. amt

$$x'(t) = r_i c_i - r_o c_o$$

$$\begin{cases} x'(t) = rc - r\dfrac{x}{V} \\ x(0) = 5cV \end{cases}$$

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \dfrac{km^3}{y}$

b) Solve the IVP, and then answer the question.

$$\begin{cases} x'(t) + \dfrac{r}{V}x = rc \\ x(0) = 5cV \end{cases}$$

$$\begin{cases} x' + ax = b \\ x(0) = x_0 \end{cases}$$

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}$$

when will $\boxed{\dfrac{x(t)}{V} = 2c\ ?}$

crib soln $a = \dfrac{r}{V}$

$b = rc$

$\dfrac{b}{a} = \dfrac{rc}{r/V} = cV$

$$2\cancel{c}V = x(t) = \cancel{c}V + (5\cancel{c}V - \cancel{c}V)e^{-\frac{r}{V}t}$$

$$2 = 1 + 4e^{-\frac{r}{V}t}$$

$$.25 = e^{-\frac{r}{V}t} = e^{-\frac{48}{35}t}$$

$t \approx 1.9$ years

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

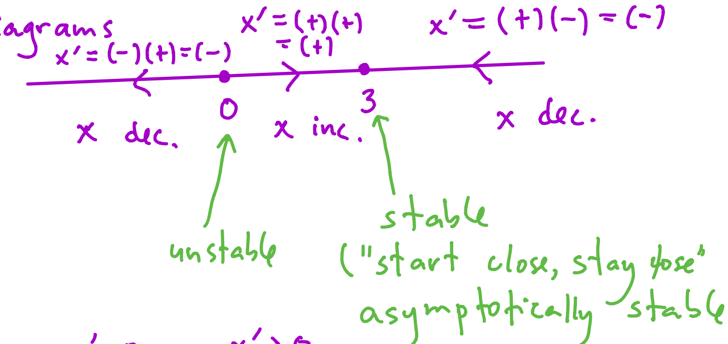
Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

phase diagrams

$x' = x(3-x)$

2a) $x'(t) = 3x - x^2$

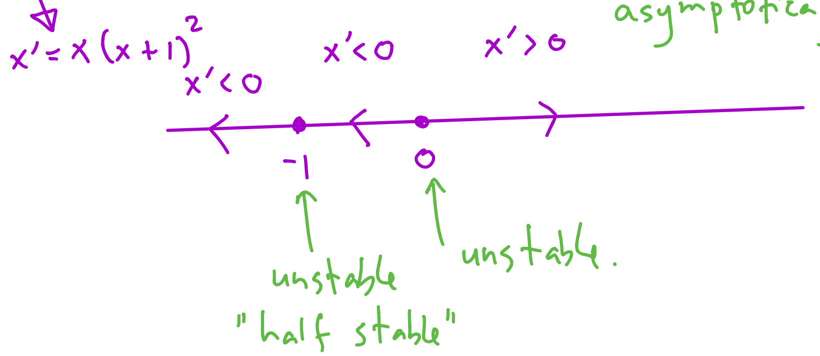
$= x(3-x)$



2b) $x'(t) = x^3 + 2x^2 + x$

2c) $x'(t) = \sin(x)$.

Friday warmup



Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$x'<0$ $\quad x'>0 \quad\quad c$

$sign(f): \; ----0+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable

case 2 $\quad sign(f): \; +++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is asymptotically stable

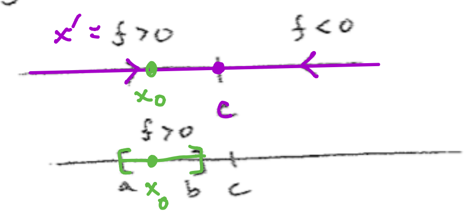
$sign(f): \; +++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable (half stable)

$sign(f): \; ----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why! "proof"

e.g. consider the second case



if x_0 is anywhere on this subinterval (except $x_0 = c$) then $\lim_{t\to\infty} x(t) = x_0$

f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

$x'(t) = f(x)$ $\quad$ (extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$

warm-up, with deltas!!

in time $\frac{b-a}{\delta}$

$x(t)$ will travel at least $(\delta)\left(\frac{b-a}{\delta}\right) = b-a$, or hit b.

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t-c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

check $\quad \frac{d}{dt} X(t) = \frac{d}{dt} x(t-c) \underset{\text{chain rule}}{=} x'(t-c) \underbrace{\frac{d}{dt}(t-c)}_{1} \underset{\text{DE}}{=} f(x(t-c)) = f(X(t))$

horizontal translations of solution graphs, are also solution graphs

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic: $P'(t) = -a\,P^2 + b\,P$ $= kP(M-P)$

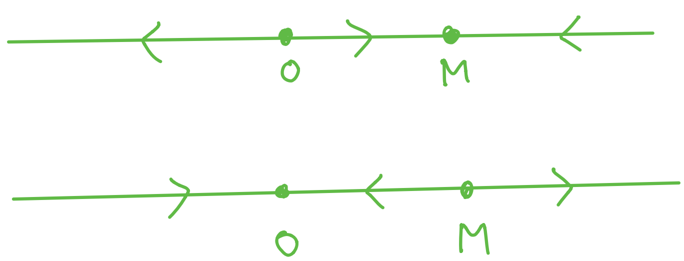
Doomsday-extinction: $Q'(t) = a\,Q^2 - b\,Q$ $= kQ(Q-M)$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q-M).$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

$P' = (+)(-) = (-)$



$P' = kP(M-P)$

$Q' = kQ(Q-M)$

Math 2280-001
Fri Jan 25
2.2 Autonomous differential equations, with applications; 2.3 improved velocity models

Announcements:
- Add 1.5.38 to this week's HW. (I'll update assignment.)
  input - output
- We'll finish 2.2, 1.5 today. Cover 2.3 Monday.
- Sections 2.4 - 2.6 are about numerical methods for solving DE's MATLAB

Warm-up Exercise: a) If a car's velocity is at least 10 miles/hour what's the longest time it could take for the car to travel 5 miles? $\frac{1}{2}$ hour.

b) Draw the phase diagram for the DE
$$x'(t) = \sin x$$
Indicate stability of equilibrium points

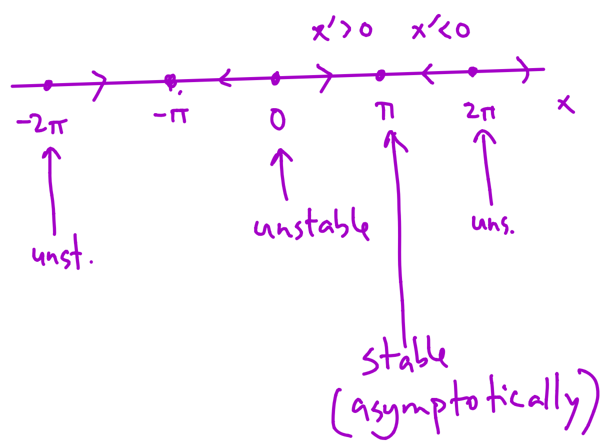
a) .5 hour Common sense & Calculus : x(t) = position fcn.

$$x(.5) - x(0) = \int_0^{.5} x'(t)\,dt \geq \int_0^{.5} 10\,dt = 10(.5) = 5$$

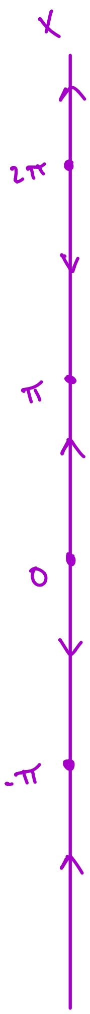
FTC $\quad x'(t) \geq 10$

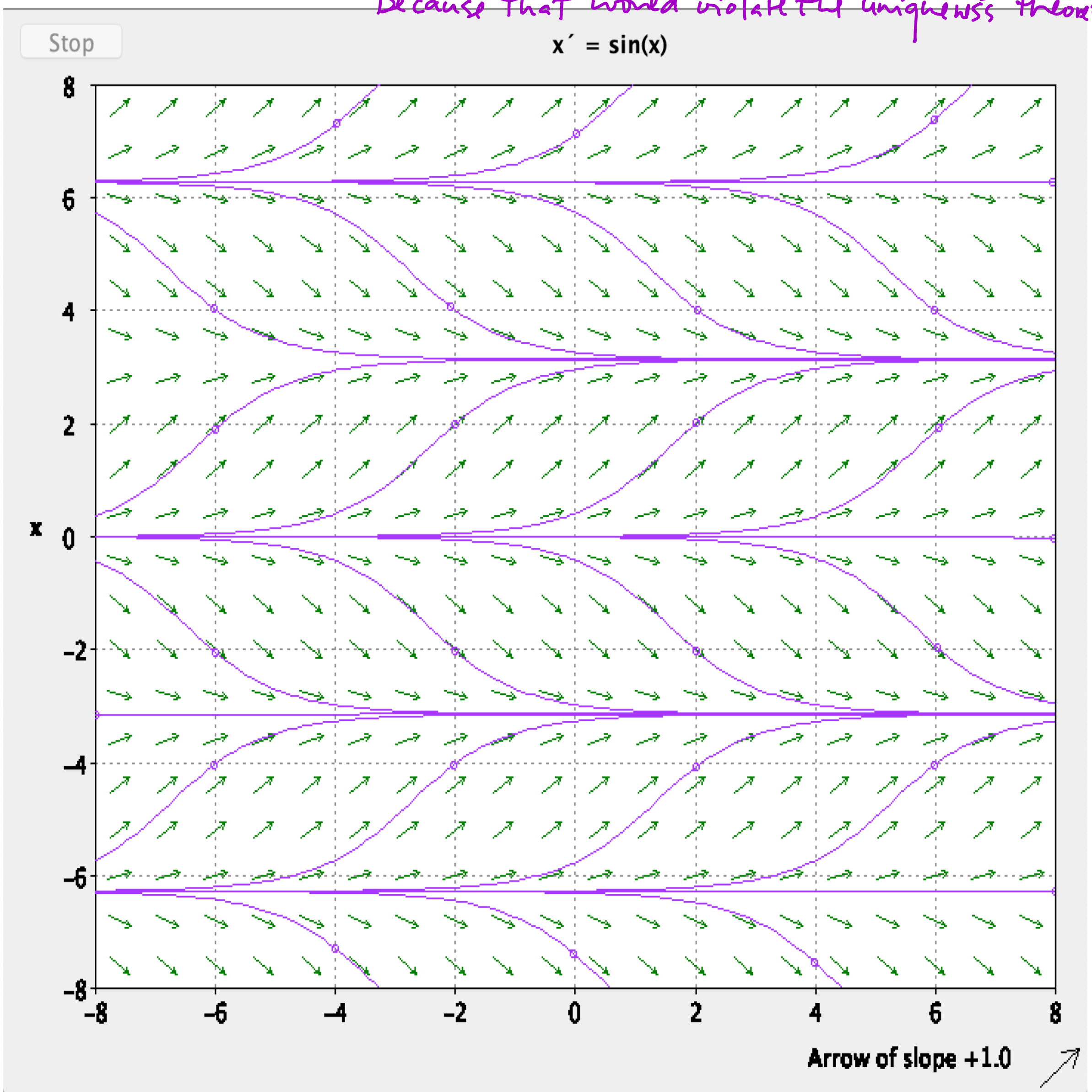
$x'(t) = \sin x$

b)



no 2 different graphs can ever touch, say at (t_0, x_0).
Because that would violate the uniqueness theorem





notice horiz translations of soltn graphs are also solut. graphs

- Recall that on Wednesday we discussed the following important concepts:
  - * Autonomous first order DE
  - * equilibrium solutions for autonomous DE's
  - * stability at equilibrium points.

<u>Further application:</u> (related to parts of a "~~yeast~~ bioreactor" homework problem for next week) harvesting a logistic population...text p.89-91 (or, why do fisheries sometimes seem to die out "suddenly"?)
Consider the DE

$$P'(t) = \underbrace{a\,P - b\,P^2}_{\text{logistic}} \; \underbrace{- \,h}_{\text{constant rate harvesting}} .$$

Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

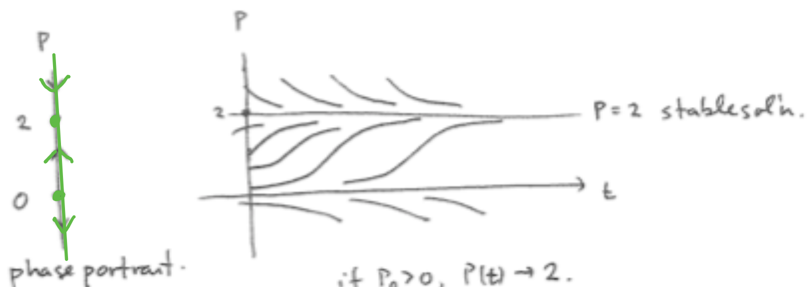
For computational ease we will assume $a = 2,\ b = 1$. (One could actually change units of population and time to reduce to this case.)

$$P'(t) = 2P - P^2 - h$$

for computational simplicity take $a = 2,\ b = 1$

<u>Case 0</u> no harvesting

$$P'(t) = 2P - P^2$$
$$P'(t) = P(2 - P)$$



if $P_0 > 0$, $P(t) \to 2$.
(units could be millions of fish, e.g.)

with harvesting: $h > 0$, small

$$P'(t) = 2P - P^2 - h$$
$$= -(P^2 - 2P + h)$$
$$= -(P - P_1)(P - P_2)$$
$$P_1, P_2 = \frac{2 \pm \sqrt{4 - 4h}}{2}$$
$$= 1 \pm \sqrt{1 - h}$$

<u>Case 1</u>: subcritical harvesting

$$0 < h < 1$$

Case 2: Critical harvesting

$h = 1$

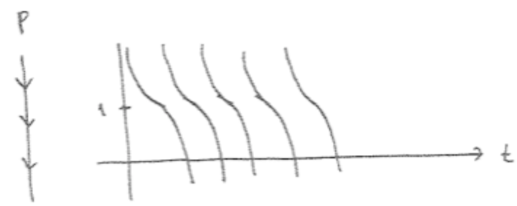
$P'(t) = -(P-1)^2$

Case 3: Over harvesting

$h > 1$

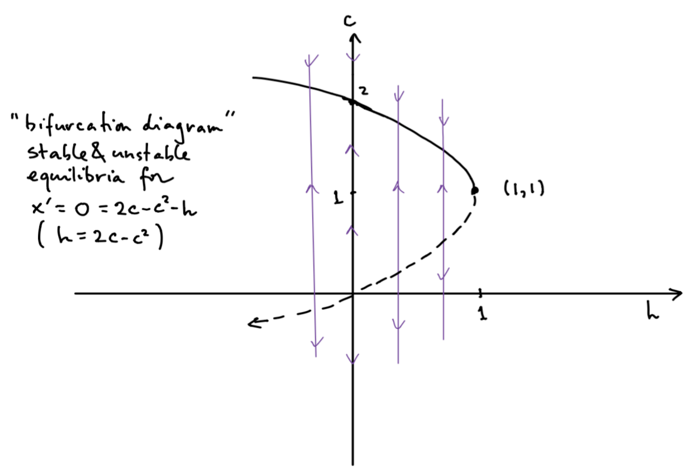
complex roots.

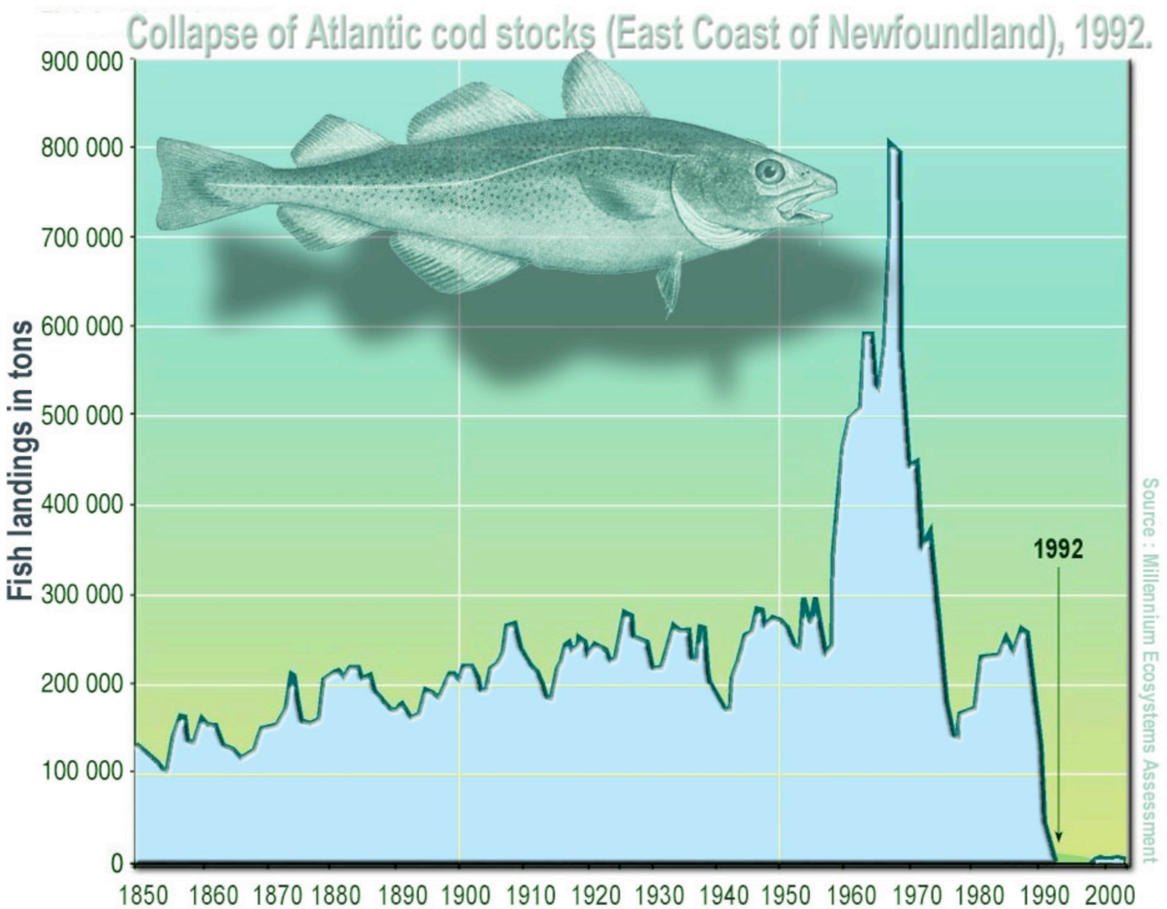
$$P'(t) = -(P^2 - 2P + h)$$
$$= -[(P-1)^2 + (h-1)]$$
$$< 0.$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.




Collapse of Atlantic cod stocks (East Coast of Newfoundland), 1992.
Fish landings in tons
900 000
800 000
700 000
600 000
500 000
400 000
300 000
200 000
100 000
0
1850
1860
1870
1880
1890
1900
1910
1920
1930
1940
1950
1960
1970
1980
1990
2000
1992
Source : Millennium Ecosystems Assessment