Wed Jan 23
2.2 phase portrait analysis and applications

Announcements:

Warm-up Exercise: Solve the IVP for $x(t)$

$$\begin{cases} x'(t) = x(x-1) = x^2 - x \\ x(0) = 2 \end{cases}$$

this is a "doomsday-extinction" population model

Hint: use partial fractions $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$

What happens as t increases from zero?

$$\int \frac{dx}{x(x-1)} = \int dt$$

$$\int \frac{1}{x-1} - \frac{1}{x}\, dx = \int dt$$

$$\ln|x-1| - \ln|x| = t + C_1$$

$$\ln\left|\frac{x-1}{x}\right| = t + C_1$$

$e \qquad e$

$$\left|\frac{x-1}{x}\right| = e^t e^{C_1}$$

$$\frac{x-1}{x} = C e^t \qquad C = e^{C_1} \text{ or } -e^{C_1}$$

$$x(0) = 2: \ \tfrac{1}{2} = C$$

$$\frac{x-1}{x} = \tfrac{1}{2} e^t$$

$x:$

$$x - 1 = \tfrac{1}{2} e^t x$$

$$x - \tfrac{1}{2} e^t x = 1$$

$$x\left(1 - \tfrac{1}{2} e^t\right) = 1 \implies \boxed{x(t) = \frac{1}{1 - \frac{1}{2} e^t}}$$

"Doomsday"

vert. asymp on graph when $1 - \frac{1}{2} e^t = 0$

$$1 = \tfrac{1}{2} e^t$$

$$2 = e^t \qquad \boxed{t = \ln 2}$$

$$x(0) = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \checkmark$$

- adj for things that only depend on themselves

2.2: Autonomous Differential Equations.

Recall, that a general first order DE for $x = x(t)$ is written in standard form as

$$x' = f(t, x) \ ,$$

which is shorthand for $x'(t) = f(t, x(t))$.

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \ . \qquad x'(t) = f(x(t))$$

Example: The logistic DE, $P' = kP(M - P)$ is an autonomous differential equation for $P(t)$.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3x - x^2 = x(3-x)$ equil soltns $x(t) \equiv 0,\ x(t) \equiv 3$ or for short $x = 0, 3$

1b) $x'(t) = x^3 + 2x^2 + x = x(x^2+2x+1) = x(x+1)^2$ equil soltns $x = -1, 0$

1c) $x'(t) = \sin(x)$. $x \equiv k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (for short $k \in \mathbb{Z}$) ↑ integers.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

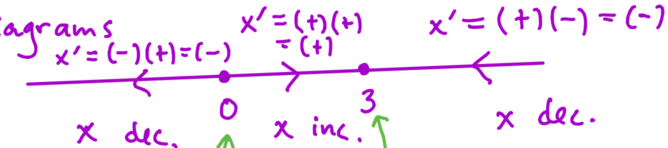
Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

phase diagrams

2a) $x'(t) = 3x - x^2$

$= x(3-x)$

$x' = x(3-x)$



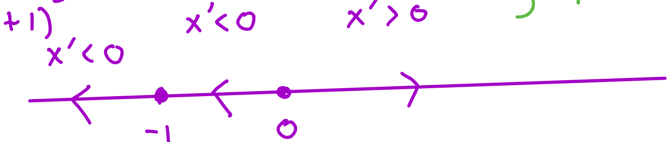
unstable

stable ("start close, stay close")
asymptotically stable.

2b) $x'(t) = x^3 + 2x^2 + x$

$x' = x(x+1)^2$

2c) $x'(t) = \sin(x)$.



unstable
"half stable"

unstable.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

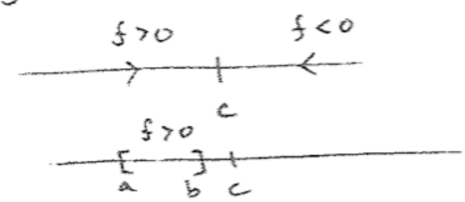
with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$$sign(f): \ ----0+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable}$$
$$sign(f): \ +++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is asymptotically stable}$$
$$sign(f): \ +++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$
$$sign(f): \ ----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ■

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t-c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic: $P'(t) = -a P^2 + b P$

Doomsday-extinction: $Q'(t) = a Q^2 - b Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\, Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\, Q^2$ and $-b\, Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\, Q\left(Q - \frac{b}{a}\right) = k\, Q(Q - M).$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M - P)$$

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q - M)\ .$$

Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}\ .$$

Exercise 5: Use your formula from the previous exercise or work the separable DE from scratch, to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x-1)$$
$$x(0) = 2\ .$$

Warmup.

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

phase diagram

constant solutions $x(t) \equiv C$

$(x'(t) \equiv 0)$

$x \equiv 0,\ x \equiv 1.$