Math 2280-002
Week 2 concepts and homework, due Wednesday January 23 at the start of class

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Wednesday quiz will be drawn from all of these concepts and from these or related problems.

1.5: **33**, **34**, **36**

2.1: 1, **3**, **6**, **10**, **12**, **23**, 33 (In 23, notice this is just a logistic DE, so you may use the general solution formula.)

2.2: 5, 7, 9, 11

**w2.1)** (section 1.4 Newton's law of cooling modeling) As part of the summer job at a restaurant, you learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. However the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at $100^o$ $C$, and the fridge was not powerful enough to accomodate a big pot of soup if it was any warmer than $20^o$ $C$.) Suppose that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at $5^o$ $C$) and stirring occasionally, you could bring the temperature of the soup to $60^o$ $C$ in 10 minutes. How long before closing time should the soup be ready so that you could put it in the fridge and leave on time?

**w2.2)** (section 1.5 modeling) A 25-year-old woman accepts an actuarial position with a starting salary of *\$70,000* per year. her salary $S(t)$ increases exponentially at a continuous rate of *5 %* per year, so that $S(t) = 70\, e^{.05\, t}$ thousand dollars per year, after t years. To save for retirement she deposits *10 %* of her salary continuously into a retirement account, which accumulates interest at an annual rate of *4 %* per year. Let $A(t)$ be the amount in the retirement account after t years, with $A(0) = 0$ thousands of dollars at the time she begins her new job.
**a)** Estimate the change $\Delta A$ in terms of $\Delta t$ to derive the differential equation for $A(t)$.
**b)** Compute the amount of money she will have in her retirement account if she retires at age 67.

**w2.3** This is a continuation of 2.1.23. Create a dfield plot for the logistic differential equation in this problem, say with $0 \le t \le 2$ and $0 \le x \le 150$, and verify that your answer to 2.1.23b is consistent with the dfield prediction, by adding the IVP solution to the dfield plot and verifying with your cursor that the $t$-value at which the solution graph passes through the horizontal line $x = 100$ agrees with your answer to 23b.

↓ postponed.

**w2.4)** Consider the differential equation

$$\frac{dx}{dt} = x^4 - 9\,x^2 .$$

**a)** Find the equilibria; draw the phase portrait;
**b)** classify the equilbria as stable, asymptotically stable, or unstable (possibly one-sided stable);
**c)** use dfield to sketch the slope field and representative solution graphs, including the graphs of the equilibrium solutions, to verify your phase portrait analysis. Include this plot in your homework.

Math 2280-001
Week 3: Jan 22-25, sections 2.1-2.3

Tues Jan 22
Finish improved population models 2.1, and discuss input-output applications 1.5.

Announcements:
- Office hours for 2-2:50 LCB 204
- 2.2 HW (very last problem i.e.) postponed 'til next week)

first 10 minutes – pretend it's a group quiz

Warm-up Exercise: Set up the DE from w2.2 HW (on handout)
(The retirement account problem)

Hint: In time $\Delta t$ (e.g. a week)

$$\underbrace{\Delta A}_{\text{change in acct}} = \text{salary contribution} + \text{interest contribution}$$

units help!!: $\$ = \quad + \quad$

Tuesday warm-up: Set up the DE from w2.2a:

**w2.1)** (section 1.4 Newton's law of cooling modeling) As part of the summer job at a restaurant, you learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. However the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at $100^o$ $C$, and the fridge was not powerful enough to accomodate a big pot of soup if it was any warmer than $20^o$ $C$.) Suppose that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at $5^o$ $C$) and stirring occasionally, you could bring the temperature of the soup to $60^o$ $C$ in 10 minutes. How long before closing time should the soup be ready so that you could put it in the fridge and leave on time?

**w2.2)** (section 1.5 modeling) A 25-year-old woman accepts an actuarial position with a starting salary of $\$70,000$ per year. her salary $S(t)$ increases exponentially at a continuous rate of $5\ \%$ per year, so that $S(t) = 70\, e^{.05\, t}$ thousand dollars per year, after t years. To save for retirement she deposits $10\ \%$ of her salary continuously into a retirement account, which accumulates interest at an annual rate of $4\ \%$ per year. Let $A(t)$ be the amount in the retirement account after t years, with $A(0) = 0$ thousands of dollars at the time she begins her new job.

**a)** Estimate the change $\Delta A$ in terms of $\Delta t$ to derive the differential equation for $A(t)$.

**b)** Compute the amount of money she will have in her retirement account if she retires at age 67.

in time $\Delta t$, $\overbrace{\Delta A}^{\text{change in } A}$ = part from salary + part from interest

units: $\$ \quad = \quad .1\ S(t)\,\Delta t \qquad .04\ A(t)\ \Delta t$

$(\$/\cancel{\text{year}})(\cancel{\text{year}}) \qquad \%/\cancel{\text{year}} \quad \$ \quad \cancel{\text{year}}$

$$\Delta A \cong .1\ S(t)\Delta t + .04\ A(t)\Delta t$$

$\div \Delta t$

let $\Delta t \to 0$

$$A'(t) = .1\ \underbrace{S(t)}_{} + .04\ A(t) \qquad \left(S(t) = 70e^{.05t}\right)$$

$$\begin{cases} A' = 7e^{.05t} + .04A \\ A(0) = 0 \end{cases}$$

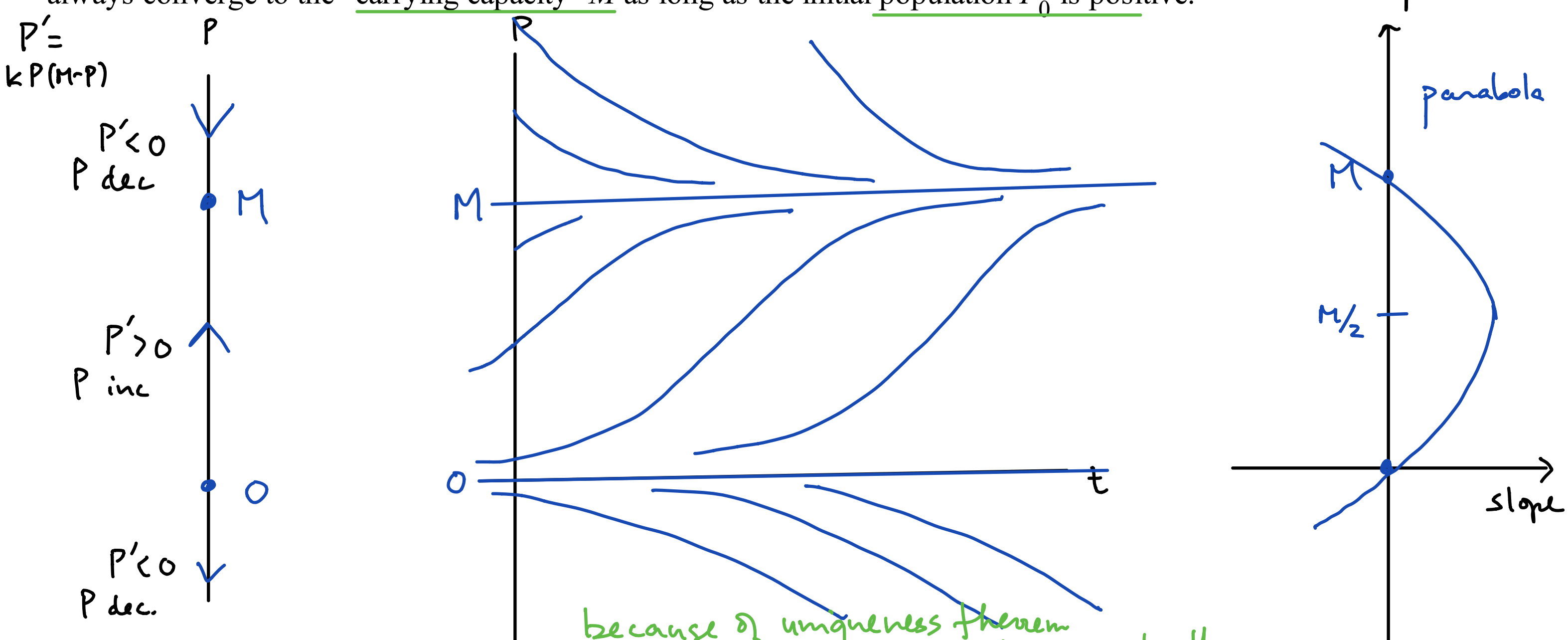
linear form:

$$\begin{cases} A' - .04A = 7e^{.05t} \\ A(0) = 0 \end{cases}$$

Friday recap: We discussed population models that can be more effective than the exponential growth/decay model in certain applications, in particular the logistic growth equation,

$$P'(t) = k\,P\,(M - P)$$

where $k$, $M$ are positive constants. By analyzing the slope field and a compressed 1-dimensional *phase diagram* related to the slope field, we deduced that it was likely that solutions to the logistic DE IVP's always converge to the "carrying capacity" $M$ as long as the initial population $P_0$ is positive.




phase diagram

because of uniqueness theorem none of solution graphs can touch each other (since $f(t,P) = kP(M-P)$ sats theorem conds)

sideways graph of slope fcn $f(P) = kP(M-P)$

Then we used partial fractions and separation of variables to solve the IVP

$$\begin{cases} P'(t) = k\,P\,(M - P) \\ P(0) = P_0, \end{cases}$$

finding that the solution

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}$$

is consistent with our predictions based on the slope field and the phase diagram.

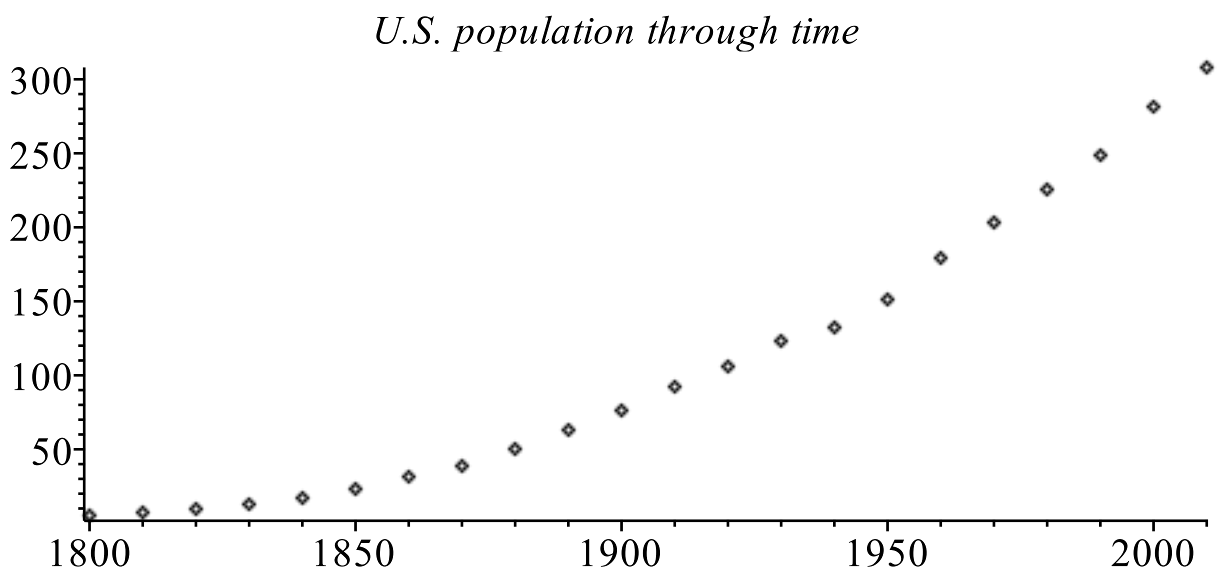
as long $P_0 > 0$

$$\lim_{t \to \infty} P(t) = \frac{M\cancel{P_0}}{0 + \cancel{P_0}} = M \checkmark$$

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

$P_0, r$ $t=0$ $t=100$

**Exponential Model:** For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for $P_0$ and $r$ :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \qquad (1)$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \qquad (2)$$

```
>
```

$P_0$ $M$, $k$ $P(50)$ $P(100)$

**Logistic Model:** We get $P_0$ from 1800, and use the 1850 and 1900 data to find $k$ and $M$ :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, e^{-M\, k\, t}} \qquad (3)$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \qquad (4)$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
    #equation (11) page 84.
```

$$M := 188.12$$

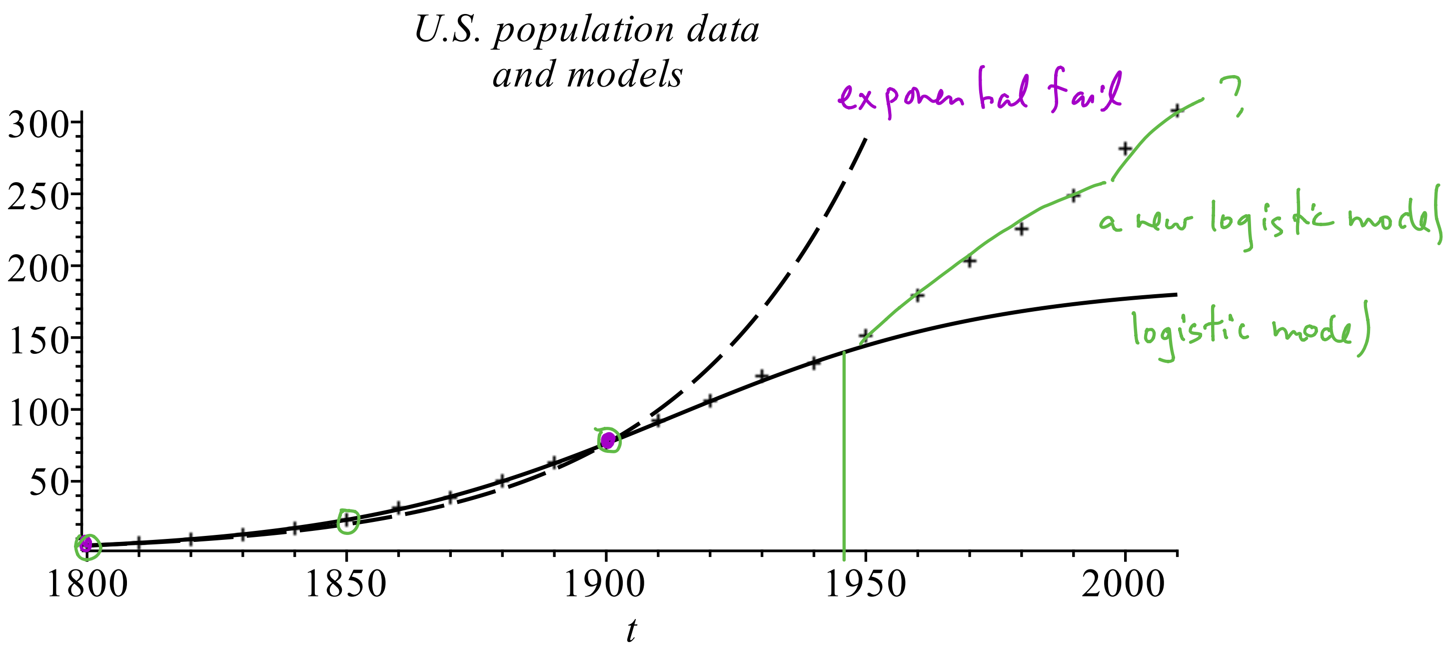
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, e^{-0.031551\, t}} \qquad (5)$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources.

```
> plot1 := plot(P1(t-1800), t = 1800 ..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t-1800), t = 1800 ..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




>

Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

baby boom, caused by
change in carrying capacity
→ agricultural
→ medicine
→ technology etc

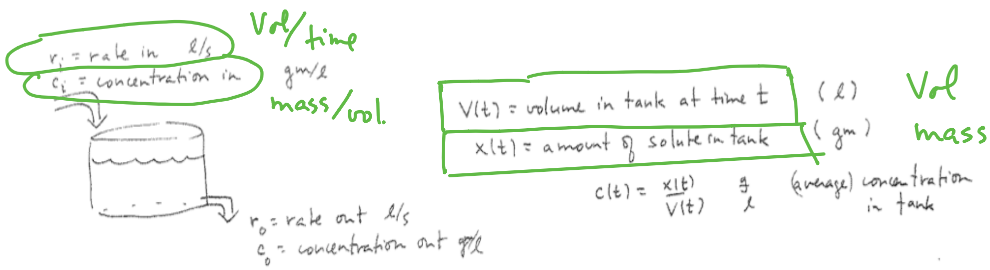
Section 1.5 modeling:

An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units $l$). Liquid flows in at a rate $r_i$ (e.g. units $\frac{l}{s}$), and with solute concentration $c_i$ (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate $r_o$, and with concentration $c_0$. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units $gm$) in the tank at time $t$, given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l} .$$

See the diagram below.




Exercise 1: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0 .$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$\boxed{x'(t) + \frac{r_o}{V} x(t) = r_i c_i}$$

spps water is flowing in at 10 l/s & flowing out at 4 l/s.
How fast is volume changing?
Ans 6 l/s
(10 − 4)

a) small time increment $\Delta t$

est $\underbrace{\Delta V}_{\text{units vol.}} \cong \underbrace{r_i}_{\text{vol}/\cancel{\text{time}}} \underbrace{\Delta t}_{\cancel{\text{time}}} - r_o \Delta t$ $\quad \div \Delta t$: $\frac{\Delta V}{\Delta t} \approx r_i - r_o$

what came in $\qquad$ what left

$\Delta t \to 0 \quad V'(t) = r_i - r_o$

b) small time inc $\Delta t$ :

$\underbrace{\Delta x}_{\text{mass}}$ = amt came in − amt left

$= \underbrace{\Delta V_{in}}_{\cancel{\text{Vol}}} \underbrace{c_{in}}_{\text{mass}/\cancel{\text{vol}}.} - (\Delta V_{out}) \underbrace{c_{out}}_{\text{"well mixed"}}$ $\quad \frac{x(t)}{V} \quad \frac{\text{mass}}{\text{vol.}}$

$\frac{\Delta x}{\Delta t} \approx \underbrace{r_i \cancel{\Delta t}}_{(a)} c_i - \underbrace{r_o \cancel{\Delta t}}_{(a)} \frac{x}{V}$

$\div \Delta t$, lim. $\quad x'(t) = r_i c_i - r_o c_o$

$x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)}$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration $c_i$ is also constant, then the IVP for solute amount is

$$\left.\begin{aligned} x' + a\,x &= b \\ x(0) &= x_0 \end{aligned}\right]$$

where $a$, $b$ are constants.

Exercise 2 (we did this last week as a warm-up exercise). The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

- $$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

Exercise 3 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 $km^3$, and water flows into and out of Lake Erie at a rate of 350 $km^3$ per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?


http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols $c$ for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \dfrac{km^3}{y}$

b) Solve the IVP, and then answer the question.