Fri Jan 16
2.1 improved population models : separable DE's

Announcements:

- We'll do the Friday notes today, so that the §2.1 HW is accessible over the long weekend.
- We'll discuss Wed.'s input-output modeling on Tuesday, and it's explained carefully in §1.5

'til 12:57
Warm-up Exercise: For the variable "P" and constant "M", what is the partial fractions decomposition for

$$* \quad \frac{1}{P(P-M)} = \frac{A}{P} + \frac{B}{P-M} \ ? \qquad A = -\frac{1}{M}, \quad B = \frac{1}{M} \ \checkmark$$

$$= \frac{1}{M}\left[\frac{1}{P-M} - \frac{1}{P}\right]$$

long way: $\frac{1}{P(P-M)} = \frac{A(P-M) + BP}{P(P-M)}$

$$1 = A(P-M) + BP$$

$$@\ P=0:\ 1 = A(-M) + 0 \Rightarrow A = -\frac{1}{M}$$

$$@\ P=M:\ 1 = BM \Rightarrow B = \frac{1}{M}$$

short way: "x" variable. α, β const

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\underbrace{\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)}_{\frac{(\cancel{x}-\beta)-(\cancel{x}-\alpha)}{(x-\alpha)(x-\beta)} = \frac{\alpha-\beta}{(x-\alpha)(x-\beta)}}$$

in our case

$$\frac{1}{(P-M)P} = \frac{1}{M}\left[\frac{1}{P-M} - \frac{1}{P}\right]$$

$$= \frac{\cancel{P} - (\cancel{P}-M)}{(P-M)P} = \frac{M}{(P-M)P}$$

2.1: Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

e.g. in Utah
$B(t) = 48{,}000 \frac{\text{people}}{\text{year}}$
$\beta(t) = \frac{48{,}000}{3{,}100{,}000}$ people/year/person
$= .015$ 1/year

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per *person*)

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per *person*)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t)$$
$$P'(t) = \left(\beta(t) - \delta(t)\right)P(t) .$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = \left(\beta_0 - \delta_0\right)P = k\,P .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

next simplest ...

Model 2: population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \left(\left(\beta_0 + \beta_1 P\right) - \left(\delta_0 + \delta_1 P\right)\right)P$$
$$= \left(\underbrace{\left(\beta_0 - \delta_0\right)}_{>0} + \left(\underbrace{\beta_1}_{<0} - \underbrace{\delta_1}_{<0}\right)P\right)P .$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$$P' = (b - a\,P)P$$
$$P' = b\,P - a\,P^2 \text{, or equivalently}$$
$$P' = a\,P\left(\frac{b}{a} - P\right) = k\,P(M - P).$$

logistic DE.

$k = a > 0, M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1a): Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$. Notice that the "isoclines" (curves where the slope function is constant) are horizontal lines

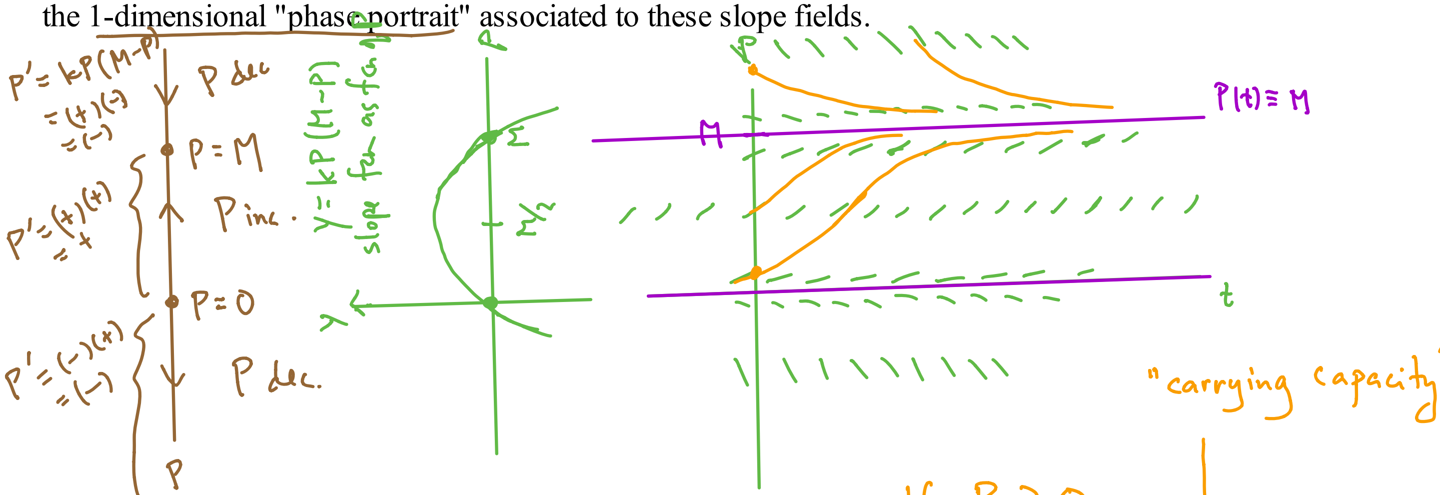
$$\frac{dP}{dt} = P' = k\,P(M - P) \qquad k > 0,\ M > 0$$

Also note that there are two constant ("equilibrium") solutions. What are they?

$$P(t) \equiv M \qquad (P' \equiv 0,\ kP(M-P) \equiv 0)$$

$$P(t) \equiv 0$$

b) Sketch the slope field and apparent solutions graphs in a qualitatively accurate way. We'll also include the 1-dimensional "phase portrait" associated to these slope fields.



If $P_0 > 0$

$$\lim_{t \to \infty} P(t) = M$$

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? Note that by the existence-uniqueness theorem, different solution graphs may never touch each other, so the time-varying solution graphs never touch the horizontal graph asymptotes.

M

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}\,.$$

Solution (we will work this out step by step in class, using the fact that the logistic DE is separable. It is not linear!!):

$$\frac{dP}{dt} = kP(M-P)$$

$$\frac{dP}{P(M-P)} = k\,dt \qquad \longleftarrow \quad P \neq 0,\ P \neq M$$

$$\int \frac{dP}{P(P-M)} = \int -k\,dt \qquad \text{for convenience.}$$

$$\Big\downarrow \text{warmup}$$

$$\int \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right) dP = \int -k\,dt$$

$$\int \left(\frac{1}{P-M} - \frac{1}{P}\right) dP = \int -Mk\,dt$$

$$\ln|P-M| - \ln|P| = -Mkt + C_1 \qquad \ln a - \ln b = \ln\frac{a}{b}$$

$$e^{\ln\left|\frac{P-M}{P}\right|} = e^{-Mkt + C_1}$$

$$\left|\frac{P-M}{P}\right| = e^{C_1}e^{-Mkt} \qquad e^{C_1} \text{ or } -e^{C_1}$$

$$\frac{P-M}{P} = \pm e^{C_1}e^{-Mkt} = C_2 e^{-Mkt} \qquad @\, t=0:\ \boxed{\frac{P_0 - M}{P_0} = C_2}$$

$$P - M = P\,C_2 e^{-Mkt}$$
$$P - PC_2e^{-Mkt} = M$$
$$P(1 - C_2e^{-Mt}) = M$$

$$P(t) = \frac{M}{1 - C_2 e^{-Mkt}} = \frac{M}{1 - \left(\frac{P_0 - M}{P_0}\right) e^{-Mkt}} \quad \frac{P_0}{P_0}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + \left(M - P_0\right)e^{-Mkt}}.$$

Notice that because $\lim_{t \to \infty} e^{-Mkt} = 0$,

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)