For your section 1.2 and 1.4 homework this week I assigned a selection of application problems. Some applications will be familiar to you from previous courses, e.g. exponential growth and Newton's Law of cooling, velocity-acceleration problems. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

<u>Toricelli's Law</u>, for draining water tanks. Refer to the figure below.
<u>Exercise 1</u>:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity ($a << A$) . Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$\boxed{v = \sqrt{2\,g\,y}}$$

g = accel. of gravity

when the water depth above the hole is $y(t)$ ~~(and g is accel of~~ gravity).

b) Use part (a) to derive the separable DE for water depth

$$\boxed{A(y)\,\frac{dy}{dt} = -k\sqrt{y} \qquad (k = a\sqrt{2\,g}\,)\,.}$$

on Wednesday!

a) Consider small Δt time increment

$\underbrace{\text{Loss of PE at top}}$ = gain in KE at bottom

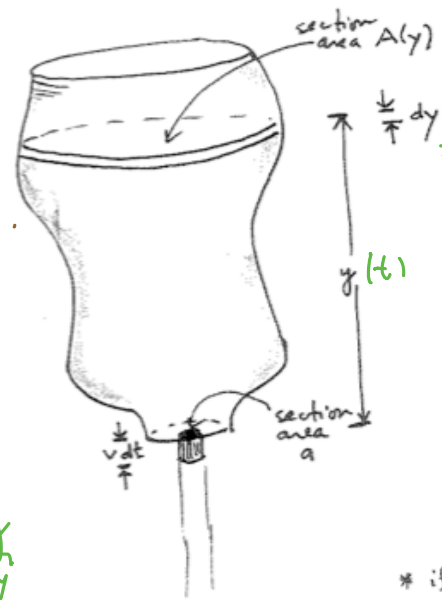
$\cancel{\rho}\,|\Delta V|\,g\,y = \frac{1}{2}\cancel{\rho}\,|\Delta V|\,v^2$

mass/vol. density; loss in vol; height

"mgh" "$\frac{1}{2}mv^2$"

$gy = \frac{1}{2}v^2 \Rightarrow v^2 = 2gy$

$v = \sqrt{2gy}$

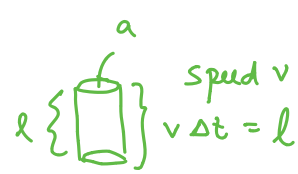


* if a<<A we may neglect the KE of the slow moving water inside the vessel.

b) figure out $\frac{dy}{dt}$:

In small Δt change Δy in ht.

loss in Vol at top = Vol lost at bottom



$A(y)\,|\Delta y| \approx a\,v\,\Delta t$

$-A(y)\,\frac{\Delta y}{\Delta t} \approx a\,v\,\cancel{\Delta t}$

$-A(y)\,y'(t) = a\,v = a\sqrt{2g}\sqrt{y}$ (a).

$$\boxed{A(y)\,y'(t) = -a\sqrt{2g}\sqrt{y}}$$

Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = h$$

$A(y) = \text{const}$

(different k than on previous page).

Exercise 4a) Solve the differential equation and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank.

$$\left.\begin{aligned} \frac{dy}{dt} &= -k\, y^{\frac{1}{2}} \\ y(0) &= 1 \end{aligned}\right\}$$

Warmup yesterday Tuesday.

Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2} t\right)^2$$

and the inverse function $t = t(y)$ is given by

$$t = \frac{2}{k}\left(1 - \sqrt{y}\right)$$

(until the tank runs empty).

$$y = \left(1 - \frac{k}{2} t\right)^2$$
$$\sqrt{y} = 1 - \frac{k}{2} t$$
$$\frac{k}{2} t = 1 - \sqrt{y}$$
$$t = \frac{2}{k}\left(1 - \sqrt{y}\right)$$

at t_1, $y = \frac{1}{2}$ so $t_1 = \frac{2}{k}(1 - \sqrt{.5})$

at t_2 $y = 0$ so $t_2 = \frac{2}{k}(1)$

$$\frac{t_2}{t_1} = \frac{\cancel{\frac{2}{k}}}{\cancel{\frac{2}{k}}(1 - \sqrt{.5})} = \frac{1}{1 - \sqrt{.5}}$$

Exercise 4b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let t_1 be the time it takes the the water to go from height 1 (full) to height 0.5 (half empty). Let t_2 be the time it takes for the water to go from height 1 (full) to height 0.0 (empty). Show that

$$t_2 = \frac{1}{1 - \sqrt{.5}}\, t_1 \approx 3.41\, t_1 .$$

Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 : # that should be enough significant digits
```

> $\frac{1}{1-\mathrm{sqrt}(.5)}$; # *the factor from previous page*

$$3.4143 \tag{3}$$

Monday $\left\{ \begin{array}{l} t_1 = 68 \text{ sec} \\ t_2 = 230 \text{ sec} \end{array} \right.$

```
> Thalf := 35;   # seconds to half-empty canteen in a previous test.
  Tpredict := 3.4143·Thalf; #prediction
```

$$Thalf := 35$$

$$Tpredict := 119.50 \tag{4}$$

$(68)(3.414) = 232$!

\>

What are possible defects in our model?

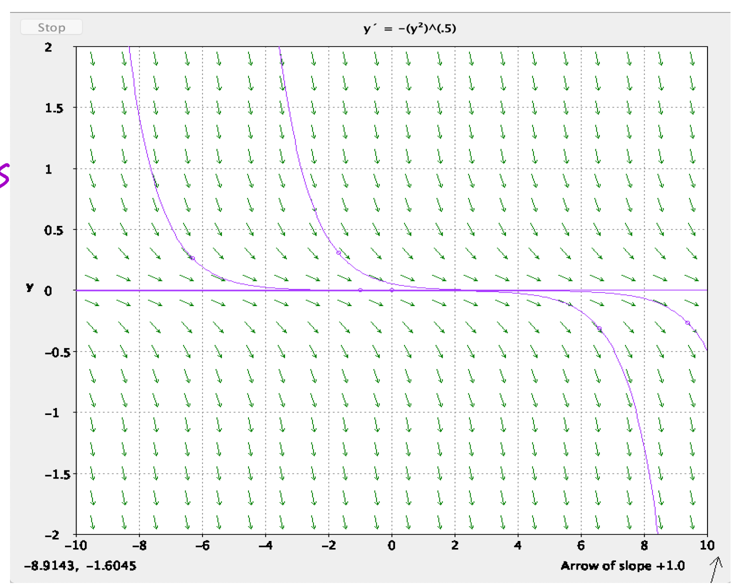
What does the existence-uniqueness theorem say about solutions to IVP's for this DE when the initial height is zero? Does this make sense? (Notice, I extended the square root function from positive to negative values by taking the square root of the absolute value, so that the existence theorem applies.)

$$\frac{dy}{dt} = -k\,|y|^{\frac{1}{2}}$$

$$y(0) = 0$$

Existence,
but not uniqueness

Makes sense: who knows when the tank emptied !



Wed Jan 16
1.5 applications of linear differential equations

Announcements:
- pick up next week's HW assignment (it's much shorter)
- quiz is separable DE + existence uniqueness thm.
- today is math modeling day
  - Torricelli §1.4
  - input-output modeling §1.5

'til 12:57

Warm-up Exercise: Solve for $x(t)$. a, b, x_0 are constants

$$\begin{cases} x' + \underbrace{a}_{P(t)} x = \underbrace{b}_{Q(t)} \\ x(0) = x_0 \end{cases}$$

$$x(t) = \frac{b}{a} + e^{-at}\left(x_0 - \frac{b}{a}\right)$$

The DE is both separable and linear. It includes exponential growth/decay, Newton's law of cooling, and lots of the input-out modeling in §1.5. We'll see it again in §2.3 & later.

often in applications, text transitions from $y(x)$ to fns $x(t)$

$$x' + \underbrace{a}_{P(t)} x = b$$

$$IF = e^{\int P(t)dt} = e^{\int a\,dt} = e^{at}$$ ← you choose your favorite antideriv. no "+C"

$$e^{at}\left[x'(t) + ax\right] = b e^{at}$$

$$\frac{d}{dt}\left[e^{at} x(t)\right] = b e^{at}$$

$$e^{at} x(t) = \int b e^{at}\,dt = \frac{b}{a} e^{at} + C$$

$\div e^{at}$ $$x(t) = \frac{b}{a} + C e^{-at}$$

find C: $x(0) = x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}$

$$\boxed{x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right) e^{-at}}$$