Tues Jan 15
1.5 linear first order differential equations.

Announcements:
- Are there any HW problems you want to discuss? Due tomorrow!
- Office hours today after class - LCB 204
- today: introduce §1.5: how to solve linear 1st order DE's
- then Toricelli (quiz tomorrow: solve a DE .. linear .. separable)

'til 12:57

Warm-up Exercise: Solve
$$\begin{cases} \frac{dy}{dt} = -k y^{1/2} \\ y(0) = 1 \end{cases}$$
k a constant.

(this is for Toricelli)

$$y(t) = \left(-\tfrac{k}{2}t + 1\right)^2 \quad \checkmark$$

$$\frac{dy}{y^{1/2}} = -k\,dt \qquad (y \neq 0 \;..\; y(t) \equiv 0 \text{ is a soln})$$

$$\int y^{-1/2}\,dy = \int -k\,dt$$

$$2y^{1/2} = -kt + C$$

$$y^{1/2} = -\tfrac{k}{2}t + C \qquad y(0)=1 \Longrightarrow 1 = 0 + C$$

$$y^{1/2} = -\tfrac{k}{2}t + 1$$

$$y = \left(-\tfrac{k}{2}t + 1\right)^2$$

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be (re)written as

$$y' + P(x)y = Q(x)$$

Exercise 1 Classify the differential equations for $y(x)$ below as linear, separable, both, or neither. Justify your answers by rewriting the DE (if necessary) so that it is in the standard format for linear or separable differential equations

$y' = f(x)g(y)$

a) $y' = -2y + 4x^2$ linear $y' + \underset{P(x)}{2}y = \underset{Q(x)}{4x^2}$ not separable

'til 1:12

b) $y' = x - y^2 + 1$ not linear
if y^2 had been y' then would've linear
not separable

c) $y' = y - x$ linear $y' - y = -x$
$P(x) = -1$, $Q(x) = -x$
not separable

d) $y' = \dfrac{6x - 3xy}{x^2 + 1}$ linear: $y' + \underbrace{\frac{3x}{x^2+1}}_{P(x)} y = \underbrace{\frac{6x}{x^2+1}}_{Q(x)}$
separable! $y' = \underbrace{\frac{3x}{x^2+1}}_{f(x)} \underbrace{(2-y)}_{g(y)}$

e) $y' = x^2 + y^2$ not linear
(if $y' = x^2 + y$, would've been)
NOPE

f) $y' = x^2 e^{x^3}$ linear: $y' + 0y = x^2 e^{x^3}$
separable $y' = (x^2 e^{x^3}) \cdot 1$

Remark: We call these *linear* differential equations because we can rewrite them as

$$L(y) = Q(x)$$

where

$$L(y) := y' + P(x)y$$

is a *linear transformation* (from 2270) !!!

$$\begin{cases} L(y_1 + y_2) = L(y_1) + L(y_2) \\ L(cy) = cL(y) \quad c \text{ const} \end{cases}$$

we shall return to these ideas in Chptr 3!

Exercise 2) (~~This is one of your homework exercises~~....)

We could explain where soltns in w1.3 came from instead:

$y' = x - y + 1$

Consider exercise 1c above, written in "linear form":

$$y' - y = -x$$

Multiply both sides by the never-zero exponential function e^{-x} to get an equivalent differential equation - in the sense that solutions to the first DE are also solutions to the second DE, and vise-verse. Why did we choose e^{-x}? It has nothing to do with the right side of this DE, and everything to do with the left side. That's explained on the next page!

$$e^{-x}(y' - y) = -x\,e^{-x}.$$

Now, if you're able to cleverly recognize the expression on the left as the derivative of a product, you'll be able to antidifferentiate both sides with respect to x, and find all solutions!

in linear form:

$$y' + y = x + 1$$

equivalent to

$$e^{x}\left[y' + y\right] = (x+1)e^{x} \qquad \text{equivalent DE}$$

magic

$$\frac{d}{dx}\left[e^{x} y(x)\right] = (x+1)e^{x} \quad !$$

$$\left(= e^{x} y + e^{x} y', \text{ by product rule}\right)$$

int. wrt x:

$$e^{x} y(x) = \int \underbrace{(x+1)}_{u}\underbrace{e^{x}\,dx}_{dv} = uv - \int v\,du$$

$$du = dx \quad v = e^{x}$$

$$= (x+1)e^{x} - \int e^{x}\,dx$$

$$e^{x} y(x) = (x+1)e^{x} - e^{x} + C$$

$\div e^{x}$:

$$y = x + 1 - 1 + Ce^{-x}$$

$$\boxed{y = x + Ce^{-x}}$$

(see w1.3)

Recall that the method for solving separable differential equations via differentials was actually using the differentiation chain rule "backwards" to antidifferentiate and find the solution functions. *The algorithm for solving linear DEs is a method to use the differentiation product rule backwards, after replacing the differential equation with its multiple by a non-zero "integrating factor" exponential function.* Here's how it goes in general!

$$y' + P(x)y = Q(x)$$

$y' + y = x$

Let $\int P(x)dx$ be <u>any</u> antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$P(x) = 1$

$\int 1\,dx = x$

IF

integrating factor

$$e^{\int P(x)dx}\left(y' + P(x)y\right) = e^{\int P(x)dx}Q(x)$$

$e^x\left[y' + y\right] = e^x x$

This makes the left side a derivative (check via product rule):

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)\ .$$

$\frac{d}{dx}\left[e^x y\right] = e^x x$

$(fg)' = f'g + fg'$

$= \left(e^{\int P(x)dx}P(x)\right)y + e^{\int P(x)dx}y'$

$= e^{\int P(x)dx}\left[P(x)y + y'\right]$

So you can antidifferentiate both sides with respect to x:

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx \quad + C.$$

$e^x y = \int x e^x\,dx$

...

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

<u>Remark</u>: If we abbreviate the function $e^{\int P(x)dx}$ by renaming it $G(x)$, then the formula for the solution $y(x)$ to the first order DE above is

$$y(x) = \frac{1}{G(x)}\int e^{\int P(x)dx}Q(x)dx + \frac{C}{G(x)}\ .$$

If x_0 is a point in any interval I for which the functions $P(x)$, $Q(x)$ are continuous, then $G(x)$ is positive and differentiable, and the formula for $y(x)$ yields a differentiable solution to the DE. By adjusting C to solve the IVP $y(x_0) = y_0$, we get a solution to the DE IVP on the entire interval. And, rewriting the DE as

$$y' = -P(x)y + Q(x)$$

we see that the existence-uniqueness theorem implies this is actually the only solution to the IVP on the interval (since $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y) = P(x)$ are both continuous). These facts would not necessarily be true for separable DE's...and we've seen how separable DE solutions may not exist or be unique on arbitrarily large intervals.

<u>Exercise 3</u> Verify that our work in Exercise 2 was following this recipe.

<u>Exercise 4:</u> Find all solutions to the linear (and separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

Hint: as you can verify below via Wolfram alpha, the general solution is $y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$. Notice that the right side of this differential equation satisfies the existence-uniqueness theorem for the rectangle which is all of $\mathbb{R}^2$, and our unique solutions exist on all of $\mathbb{R}$, $-\infty < x < \infty$ (in contrast to what can happen for general separable differential equations).

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1)$$

$(u = x^2+1)$

$$IF = e^{\frac{3}{2}\ln(x^2+1)} = \left[e^{\ln(x^2+1)}\right]^{3/2} = (x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}\left[y' + \frac{3x}{x^2+1}y\right] = \frac{6x}{x^2+1}(x^2+1)^{3/2}$$

$$\frac{d}{dx}\left[(x^2+1)^{3/2}y\right] = 6x(x^2+1)^{1/2}$$

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$\int 3u^{1/2}\,du = 2u^{3/2} = 2(x^2+1)^{3/2} + C$$

$$(x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + C$$

$\div (x^2+1)^{3/2}$

$$\boxed{y = 2 + \frac{C}{(x^2+1)^{3/2}}}$$

$$y' + P(x)y = Q(x)$$

$$\int P(x)\,dx$$

$$IF = e^{\int P(x)\,dx}$$

$$e^{\int P(x)dx}\left[y' + P(x)y\right] = e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = //$$

then antidiff

Input:

$$y'(x) = \frac{6x - 3x\,y(x)}{x^2+1}$$

Differential equation solution:

$$y(x) = \frac{c_1}{(x^2+1)^{3/2}} + 2$$