Math 2280-2
Week 2, Jan 14-18: sections 1.3-1.5, 2.1.
Mon Jan 14
Finish discussion of existence-uniqueness theorem from 1.3; Toricelli's law application from 1.4.

Announcements:

Warm-up Exercise: Get out your stopwatches, we'll do a timed experiment right at the start of class (we might even start a minute or two early.)

Review of last week:

We understand what it means for functions $y(x)$ to solve a differential equation

$$y' = f(x, y)$$

$\leftarrow$ subs $y(x)$ into DE yields a true identity

and/or an initial value problem on some interval I with $x_0 \in I$:

$$\text{IVP} \begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

We know how to find implicit and possibly explicit solutions to *separable* differential equations

$$y'(x) = f(x)g(y)$$

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx \ \ldots$$

(at least when $g(y) \neq 0$)

which extends the special case of direct antidifferentiation

$$y'(x) = f(x).$$

We understand the connection between slope fields for differential equations and graphs of solutions to initial value problems.

Because of geometric intuition based on slope fields, we expect each initial value problem for a reasonable first order differential equation to have one and only one solution, at least defined on some interval containing the intial variable value. On Friday we saw that this isn't actually always true, but there is a Theorem that explains the situation...

Here's what's going on (stated in 1.3 page 22 of text as *Theorem 1*; partly proven in Appendix A.)

Existence - uniqueness theorem for the initial value problem

Consider the IVP

$$\text{IVP}\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(a) = b \end{cases}$$

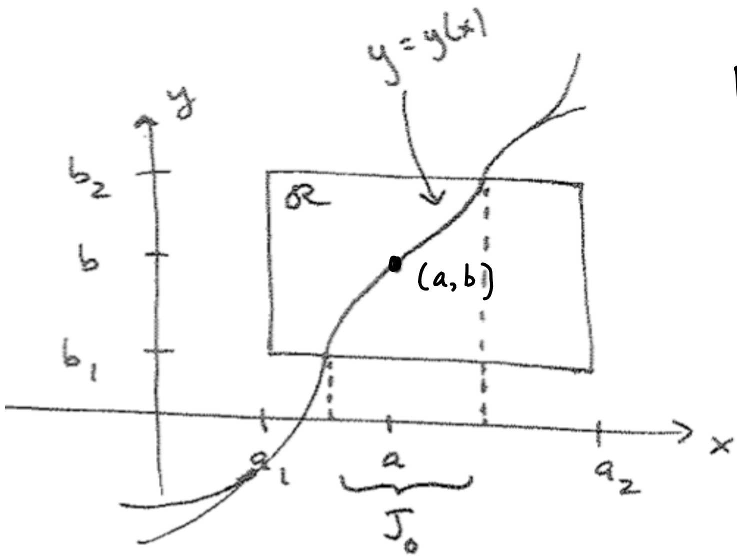
- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \le x \le a_2, b_1 \le y \le b_2$ that *you* specify in the $x-y$ plane. or $a_1 < x < a_2$, $b_1 < y < b_2$ etc.

- Existence: If $f(x, y)$ is continuous in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.

- Uniqueness: If the partial derivative function $\dfrac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

f is continuous in $\mathcal{R}$ means it's continuous at each point in $\mathcal{R}$.
f is cont. at a point (x_1, y_1) means "you can force $f(x,y)$ to be as close as you want to $f(x_1, y_1)$, by making (x,y) close enough to (x_1, y_1)

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



precise: for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$|f(x,y) - f(x_1,y_1)| < \varepsilon$$

whenever

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right\| < \delta$$

any function with reasonable formula will be continuous, except at points where e.g. denominators $= 0$ or expressions are not defined

Exercise 1 On the next page - which is our last completed page from Friday - explain how existence-uniqueness theorem applies for that initial value problem

From Friday...

Exercise 3a) Use separation of variables to solve the IVP

$$\text{IVP}\begin{cases}\dfrac{dy}{dx} = y^{\left(\frac{2}{3}\right)}\\ y(0) = 0\end{cases}$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

3c) Sketch some of these singular solutions onto the slope field below.

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx \qquad \text{Assuming } g(y) \neq 0$$

If $g(y^*) = 0$ then $y(x) = y^*$ is a sol'n. (y^* const)

$y'(x) = 0$

$f(x)g(y) = f(x)g(y^*) = f(x)\cdot 0 = 0$

Monday warmup

3a). We got sol'n's to the DE $y(x) = \frac{1}{27}(x+C)^3$

So for IVP $y_1(x) = \frac{1}{27}x^3$

3b) also noticed $y_2(x) \equiv 0$ solves IVP. keyed into that via sep. of variables

∞'ly many sol'ns

$$y_3(x) = \begin{cases} 0 & -3 \le x \le 2 \quad \text{"2" was arbitrary} \\ \frac{1}{27}(x-2)^3 & x \ge 2 \\ \frac{1}{27}(x+3)^3 & x < -3 \quad \text{"-3" was arbitrary}\end{cases}$$

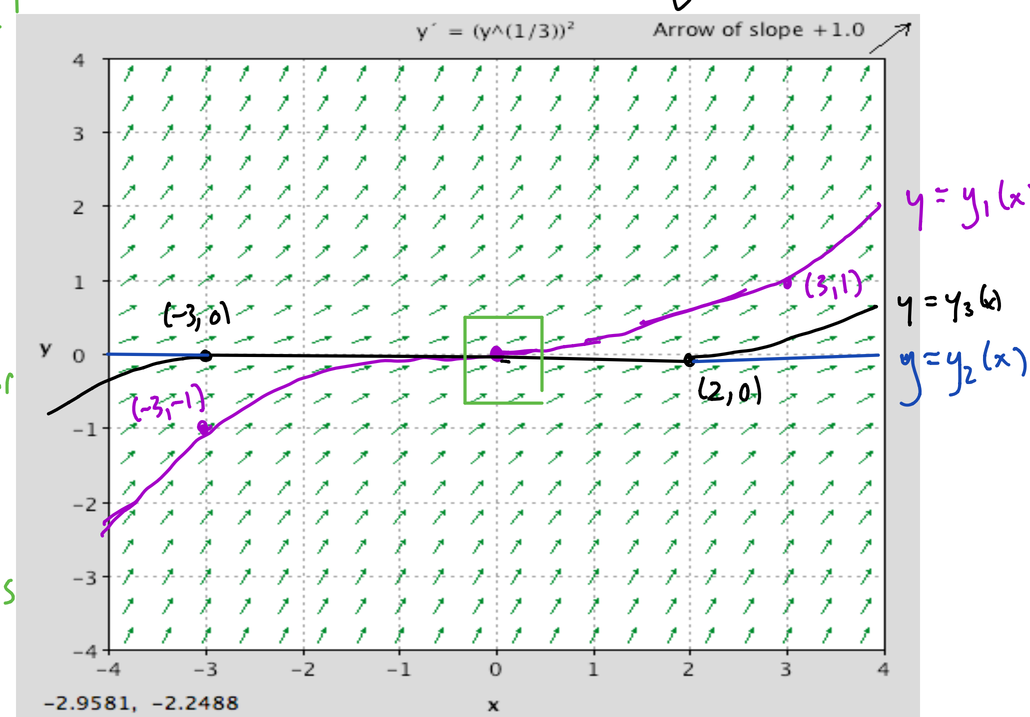
slope function $f(x,y) = y^{2/3} = (\sqrt[3]{y})^2$ is continuous in all of $\mathbb{R}^2 = \mathbb{R}$

So solutions to IVP exist

but.

$$\frac{\partial f}{\partial y} = \frac{2}{3}y^{-1/3} = \frac{2}{3}\frac{1}{\sqrt[3]{y}}$$

is continuous on all of $\mathbb{R}^2$ except when $y = 0$, i.e. x-axis



No matter how small a coord rectangle containing $(0,0)$ inside of it, you choose $\frac{\partial f}{\partial y}$ will not be continuous in that rect., because rectangle contains part of x-axis.

that's why uniqueness thm couldn't guarantee unique solutions

Exercise 2 (A slight variation on the preceding one. Also, one of your homework problems is similar.) Consider the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(3) = 8.$$

a) Does the IVP above have a unique solution on some interval containing $x_0 = 3$, according to the existence-uniquenss theorem?

$f(x,y) = y^{2/3}$ is cont. in all of $\mathbb{R}^2$, so solutions exist

$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$ cont. except along x-axis

Let $\mathcal{R}$ be the upper half plane $-\infty < x < \infty$, $y > 0$

Thm applies so unique sol

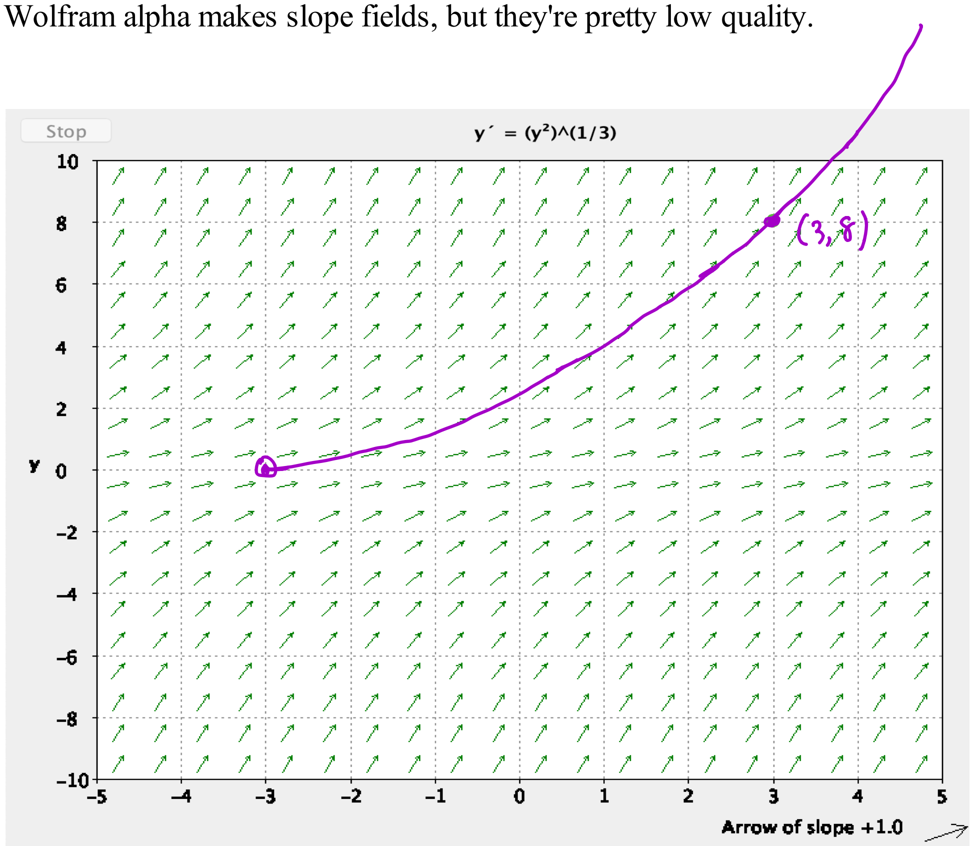
b) Find the IVP solution above, using separation of variables solutions that we found on Friday

$$\left(\frac{x+3}{3}\right)^3 = y(x) = \frac{1}{27}(x+C)^3 = \left(\frac{x+C}{3}\right)^3 \qquad y(3) = 8 = \left(\frac{3+C}{3}\right)^3$$

c) What is the largest interval on which it is the unique solution? Sketch below! What's the biggest rectangle $\mathcal{R}$ that you can specify for uniqueness?

Solt'n unique for $x > -3$

d) What happens when you solve this IVP numerically with dfield? I'll demo dfield, since I'm asking you to use it for your homework. Wolfram alpha makes slope fields, but they're pretty low quality.



Exercise 3: Do the initial value problems below always have unique solutions? Would you be able find formulas for them? (Notice two of these are NOT separable differential equations.) Can technology find formulas for the solution functions?

a)

$$\frac{dy}{dx} = y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$

$f(x,y) = (x+1)(y-3)^2$ is cont in $\mathbb{R}^2$

$\frac{\partial f}{\partial y} = (x+1)\,2(y-3)$ is cont in $\mathbb{R}^2$

$\mathcal{R} = \mathbb{R}^2$. existence-uniq.

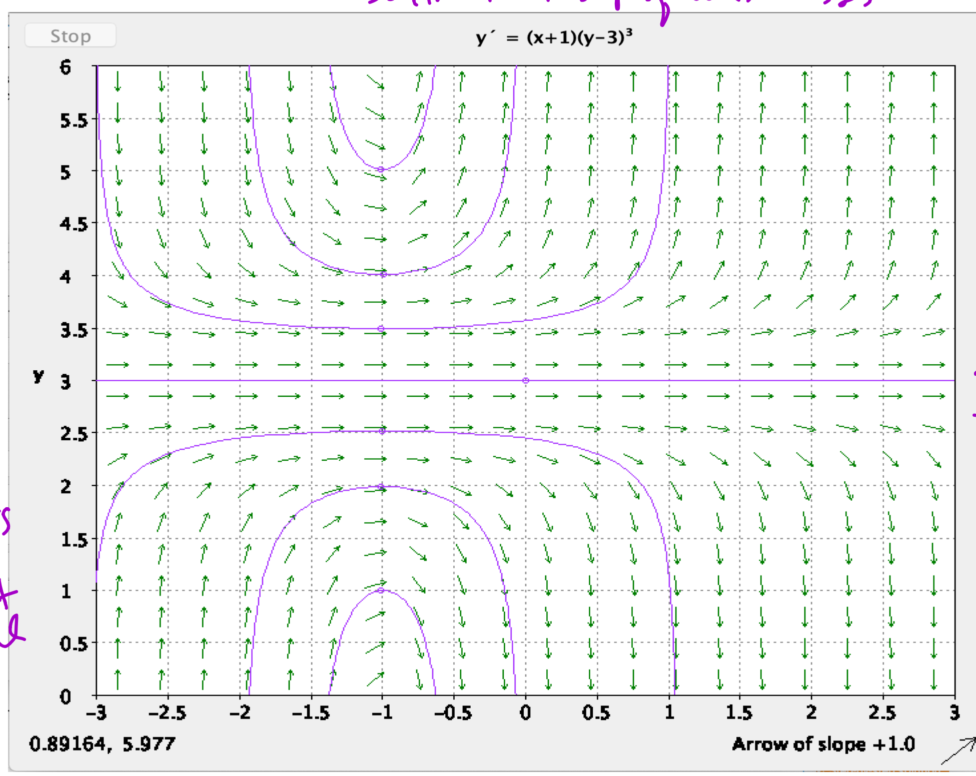
separable:

$$\frac{dy}{(y-3)^2} = (x+1)\,dx$$

$y \neq 3$

Note $y(x) \equiv 3$ is a soln that sep of vars misses

notice no other soln graph touches $y \equiv 3$ → that would violate uniqueness theorem for IVP with that touch point as the initial point



$y = 3$

I used Maple - you'd get the same results with Wolfram alpha. Which solution did technology miss?

$y(x) \equiv 3$

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> with(DEtools) :
> dsolve(y'(x) = (x + 1)·(y(x) − 3)^2, y(x));
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$$y(x) = \frac{3x^2 + 6\_C1 + 6x - 2}{x^2 + 2\_C1 + 2x} \tag{1}$$

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>
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b)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

all IVP's have unique soltns

$$\left.\begin{aligned} f(x,y) &= x^2+y^2 \\ \frac{\partial f}{\partial y} &= 2y \end{aligned}\right\} \text{both are continuous in } \mathbb{R}^2$$

> $dsolve\left(y'(x) = x^2 + y(x)^2, y(x)\right);$

$$y(x) = \frac{\left(-\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\_C1 - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\right)x}{\_C1\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)} \qquad \textbf{(2)}$$

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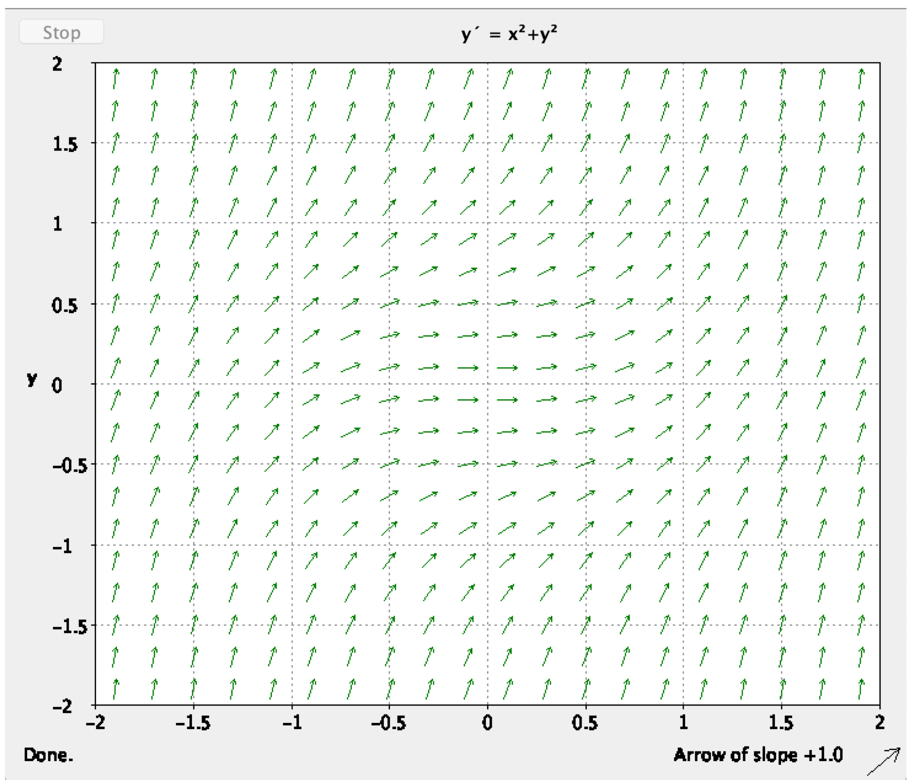

c)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

all IVP's have unique sol'ns:

$f(x,y) = x^4 + y^4$
$\frac{\partial f}{\partial y} = 4y^3$
} both are continuous in $\mathbb{R} = \mathbb{R}^2$

```
> dsolve(y'(x) = x^4 + y(x)^4, y(x));
>
```

← no formula for sol'ns!!