On Friday :

"slope fcn" only depends on x.

Exercise 1: Consider the differential equation $\boxed{\frac{dy}{dx} = x - 3}$, and then the IVP with $\boxed{y(1) = 2}$.

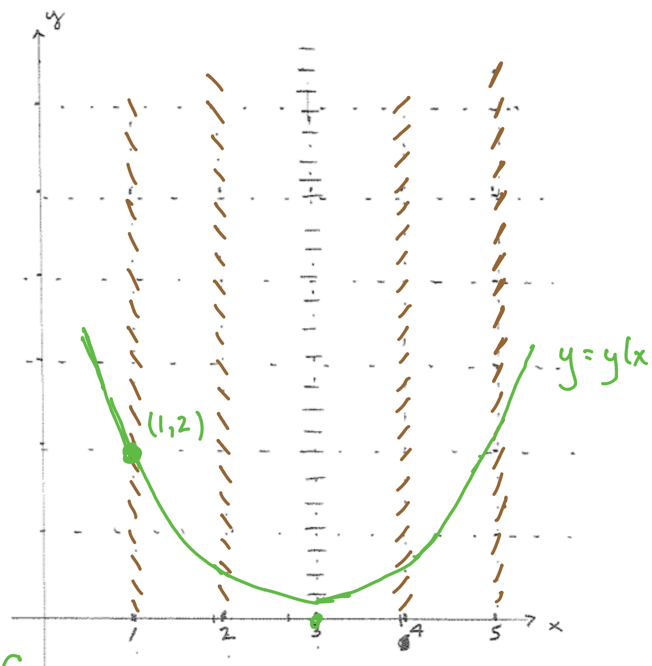
a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \leq x \leq 5, 0 \leq y \leq 6$. Notice that in this example the value of the slope field only depends on *x*, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.

b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.

c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

a)

value of slope	slope fcn $x-3$
0	$x-3=0 \quad x=3$
1	$x=4$
-1	$x-3=-1 \Rightarrow x=2$
2	$x=5$
-2	$x-3=-2 \Rightarrow x=1$
	could fill in more!



c) $y'(x) = x-3$

$y(x) = \int x-3 \, dx = \frac{1}{2}x^2 - 3x + C$

$y(1) = 2 = \frac{1}{2} - 3 + C$

$2 = -2.5 + C$

$4.5 = C$

$C = \frac{9}{2}$

$y(x) = \frac{1}{2}x^2 - 3x + \frac{9}{2}$

$= \frac{1}{2}(x^2 - 6x + 9)$

$y(x) = \frac{1}{2}(x-3)^2$

parabola with vertex $(3,0)$

The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

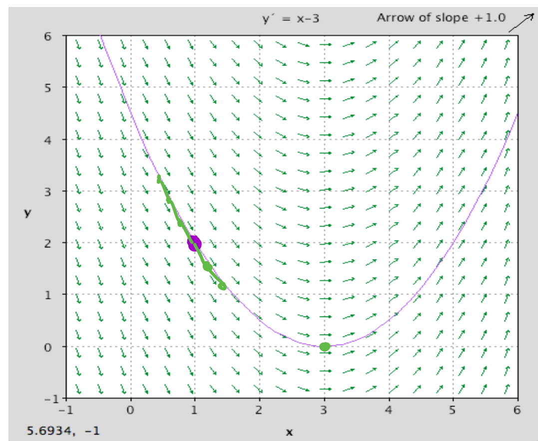
$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0), by picking a step size $\triangle x$ and then incrementing y by

$$\triangle y = f(x, y) \triangle x.$$

In this way one can *approximate* solution functions to initial value problems, and their graphs. The Java applet "dfield" (stands for "direction field", which is a synonym for slope field) uses (a more sophisticated analog of) this method to compute approximate solution graphs.

Here's a picture like the one we sketched by hand on the previous page, created by dfield.



Exercise 2: Consider the IVP

LHS RHS

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

NOT separable!! (but "linear" §1.5)

a) Check that $y(x) = x + 1 + C\,e^{x}$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.
b) Solve the IVP by choosing appropriate C.
c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?
d) Compare your work in (c) with the picture created by dfield on the next page.

a) check $y(x) = x + 1 + Ce^{x}$ solves DE

LHS $y'(x) = 1 + Ce^{x}$

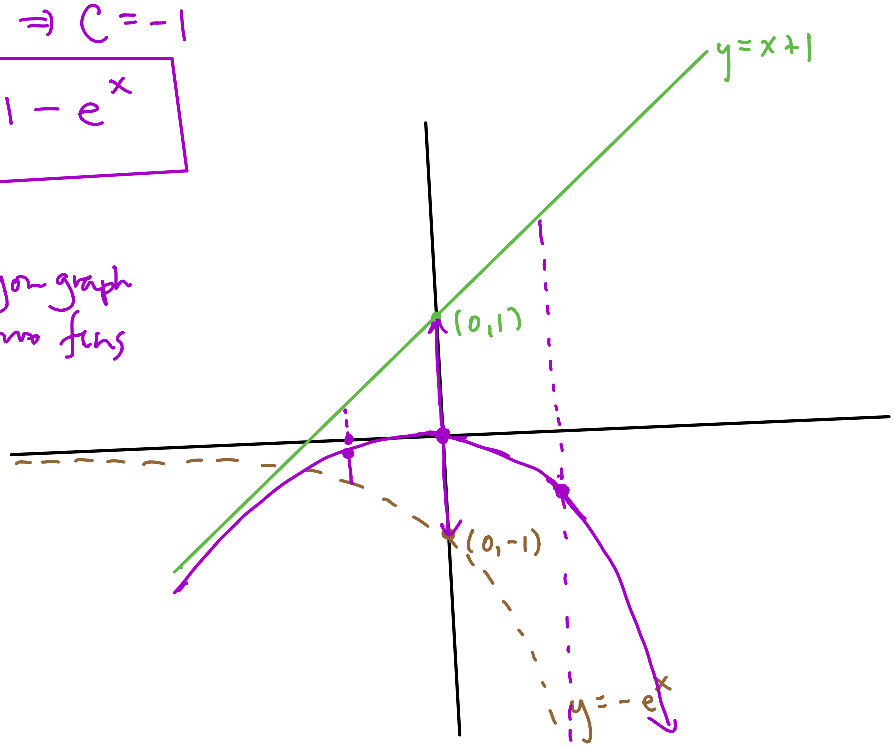
RHS $y - x = y(x) - x = \cancel{x} + 1 + Ce^{x} - \cancel{x} = 1 + Ce^{x}$

since LHS = RHS, $y(x)$ are solns

b) $y(0) = 0 = 1 + C \Rightarrow C = -1$

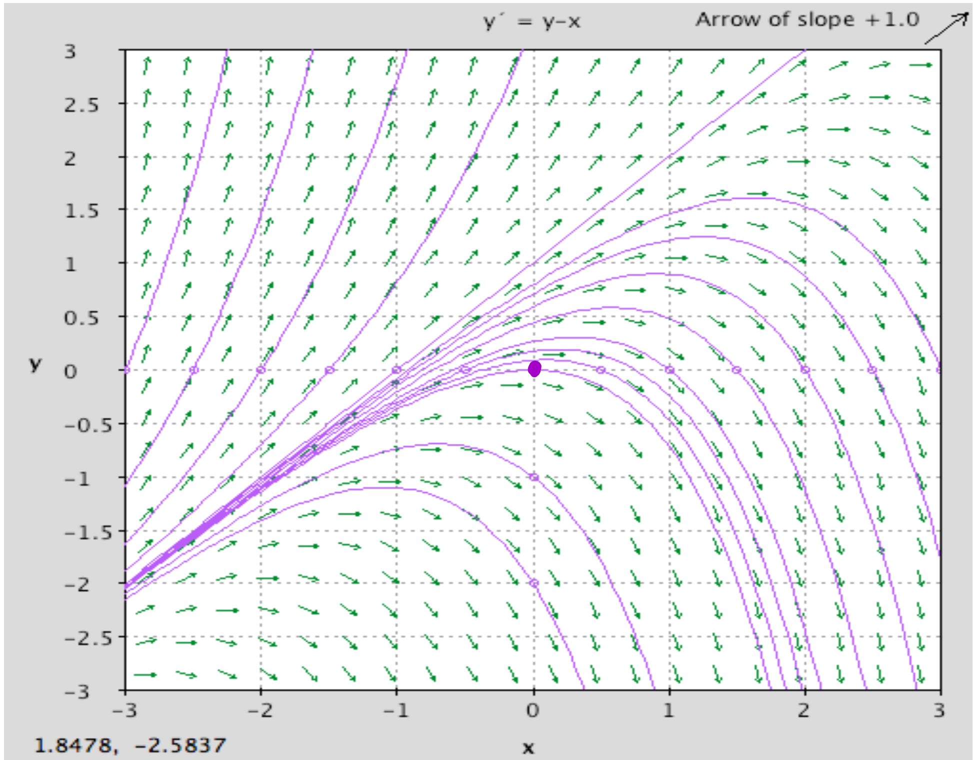
$$\boxed{y(x) = x + 1 - e^{x}}$$

review of how you graph the sum of two fns



Moral of Ex 1, 2:

knowing slope field geometry is "same" as knowing geometry of the family of solution graphs



Exercise 3a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(0) = 0$$

3b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

3c) Sketch some of these singular solutions onto the slope field below.

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx \qquad \text{Assuming } g(y) \neq 0$$

If $g(y^*) = 0$ then $y(x) = y^*$ is a sol'n. (y^* const)

$y'(x) = 0$

$f(x)g(y) = f(x)g(y^*) = f(x)\cdot 0 = 0$

warmup 3a). We got sol's to the DE $y(x) = \frac{1}{27}(x+C)^3$

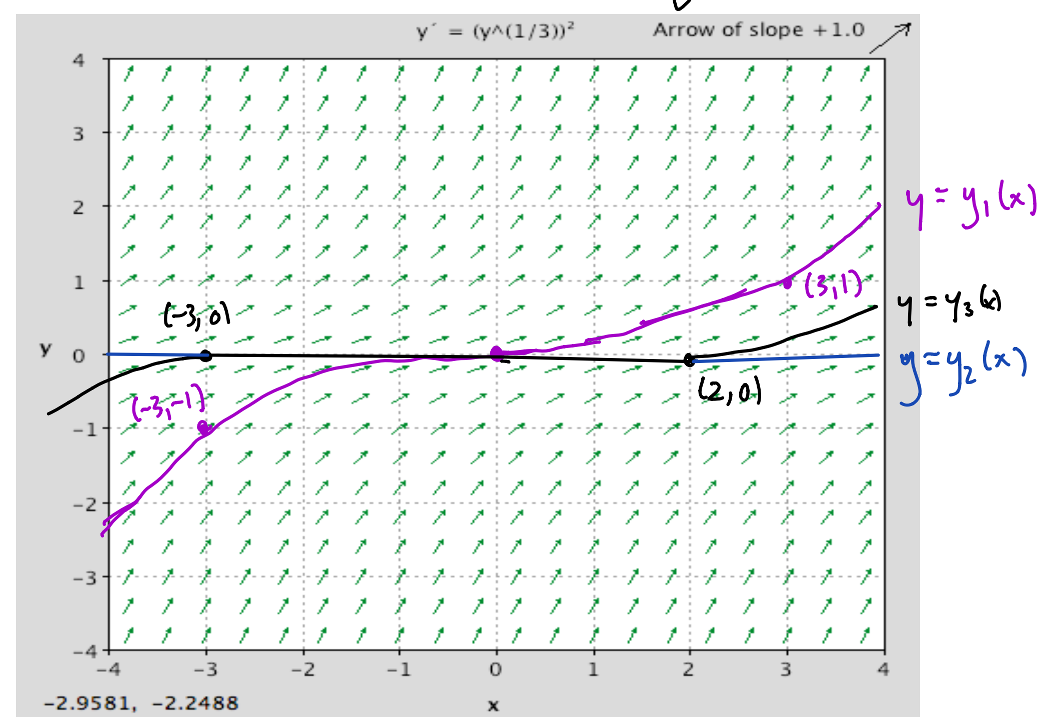
so for IVP $y_1(x) = \frac{1}{27}x^3$

3b) also noticed $y_2(x) \equiv 0$ solves IVP.

keyed into that via sep. of variables

∞'ly many sol'ns

$$y_3(x) = \begin{cases} 0 & -3 \le x \le 2 \quad \text{"2" was arbitrary} \\ \frac{1}{27}(x-2)^3 & x \geq 2 \\ \frac{1}{27}(x+3)^3 & x < -3 \quad \text{"-3" was arbitrary} \end{cases}$$

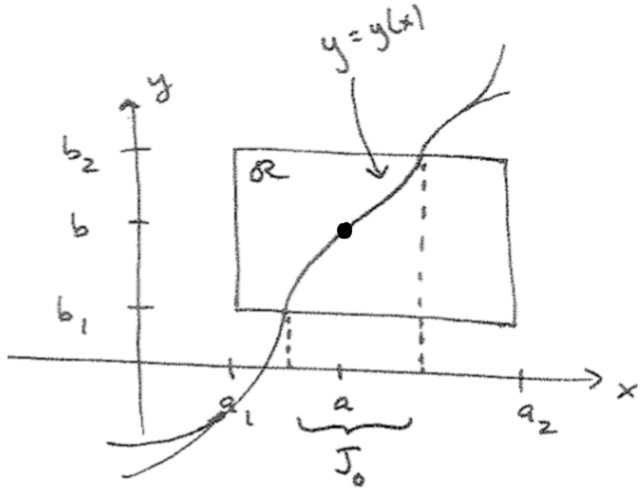


Here's what's going on (stated in 1.3 page 22 of text as *Theorem 1*; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \le x \le a_2, b_1 \le y \le b_2$ in the $x-y$ plane.
- <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 4</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Exercises 1-3 in today's notes.

Math 2280-001
Fri Jan 11
1.3-1.4 more slope fields and existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Announcements:

- CANVAS is up, Chptrs 1.1-1.5 are in "files"
- Office Hours LCB 204
  M: 10:45-11:35 a.m.
  T: 2:00-3:00 p.m.
  also available briefly after class, and by appointment.
- today do Wed notes

'til 12:57

Warm-up Exercise: Use separation of variables to solve for $y(x)$

$$\begin{cases} \frac{dy}{dx} = y^{2/3} \\ y(0) = 0 \end{cases}$$

We'll use this today.

$y^{-2/3}\, dy = 1 \cdot dx$

$\left(\frac{1}{y^{2/3}}\, dy \right)$

$\int y^{-2/3}\, dy = \int 1 \cdot dx$

$3y^{1/3} + C_1 = x + C_2$

$3y^{1/3} = x + C$

$y^{1/3} = \frac{1}{3}(x + C)$

$\boxed{y = \frac{1}{27}(x+C)^3}$

if $y(0) = 0 \Rightarrow C = 0$
gave $y_1(x)$.

assumed $y \neq 0$

Note $y_2(x) \equiv 0$ is a solution to DE & IVP
$y_2(0) = 0$ ✓
LHS $y_2'(x) = 0$
RHS $(y_2(x))^{2/3} = (0)^{2/3} = 0$
✓

a solution
$y_1(x) = \frac{1}{27}x^3$.
$y_1(0) = 0$ ✓
LHS $y_1'(x) = \frac{1}{27} 3x^2 = \frac{x^2}{9}$
RHS $\left(\frac{1}{27}x^3\right)^{2/3} = \frac{1}{9}x^2$
✓

OH OH.