Math 2280-002
Week 9-10 concepts and homework Due Wednesday March 20
sections 5.3, 6.1.

Understanding phase portraits for first order homogeneous systems of two linear differential equations, $\underline{\mathbf{x}}' = A\,\underline{\mathbf{x}}$ *in terms of the eigendata and general solutions.*
5.3 **4**, **6**, **8**, **11**, 17-22.

Finding equilibrium solutions for autonomous systems of differential equations.
6.1 1, **2**, **6**, **8**.

w9.1) This is a carry-over of problem w8.3 from last week:
a) Use the eigenvalue-eigenvector method (with complex eigenvalues) to solve the first order system initial value problem which is equivalent to the second order differential equation IVP that we discuss on Wednesday March 6 but that is in the Tuesday notes. This is the reverse procedure from class, where we use the solutions from the second order DE IVP to deduce the solution to the first order system IVP. In part b you'll verify that your answer here is consistent with our work there.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

b) Verify that the first component $x_1(t)$ of your solution to part a is indeed the solution $x(t)$ to the IVP we started with,

$$x''(t) + 2\,x'(t) + 5\,x(t) = 0$$
$$x(0) = 4$$
$$x'(0) = -4.$$

w9.2) This problem ties together our discussions in class about a number of topics: Second order differential equations can be converted into equivalent first order systems of two differential equations; if we are in the realm constant coefficient homogeneous DE's/Systems then the terminology "characteristic polynomial", "Wronskian" match up under this correspondence; in conservative systems total energy (KE+PE) is constant and so once the system is set into motion the position-velocity curve $[x(t), v(t)]$ must stay on the initial level curve of the total energy function; this can be visualized using software like pplane.

a) Consider the undamped unforced differential equation with general initial condition

$$x''(t) + 9\,x(t) = 0$$
$$x(0) = x_0$$
$$x'(0) = v_0.$$

Find the solution to this IVP in terms of x_0 and v_0

b) Without using the solution $x(t)$ to part **a** show just from the IVP above that if $x(t)$ is the solution to **a**, then $[x(t), x'(t)]^T = [x(t), v(t)]^T$ solves the first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

c) Use your result from **a** to deduce the solutions $[x_1(t), x_2(t)]^T$ for the IVP in **b.**

d) We know from previous discussions that in this undamped configuration,

$$KE + PE = \frac{1}{2} m\, v^2 + \frac{1}{2} k\, x^2 = \frac{1}{2} v^2 + \frac{9}{2} x^2$$

is constant. Compute and simplify this total energy expression for your solution to **a** and verify that, indeed,

$$\frac{1}{2} v(t)^2 + \frac{9}{2} x(t)^2 = \frac{1}{2} v_0^2 + \frac{9}{2} x_0^2.$$

So, conservation of energy and explicit computation both verify that the parametric solution curves $[x(t), v(t)]^T$ lie on ellipses that (after multiplying the identity above by $\frac{9}{2}$) can be expressed with implicit equations

$$x^2 + \frac{v^2}{9} = C, \text{ where } C = x_0^2 + \frac{v_0^2}{9}.$$

e) Use pplane to create a picture of the tangent field for the first order system of differential equations in part **b**. Include a collection of representative solution trajectories. This "phase plane" picture should illustrate your work in **d**. Hand in a printout of the result.

f) Extra credit and extra practice: Solve the first order system in **b** using the complex eigenvalue-eigenvector method and verify that the first component function of your solution $x_1(t)$ is the solution to the IVP in **a**,

w9.3a) For the first order system in **w9.1** is the origin a stable or unstable stable equilibrium point? What is the precise classification based on the descriptions of isolated critical points in section 5.3?

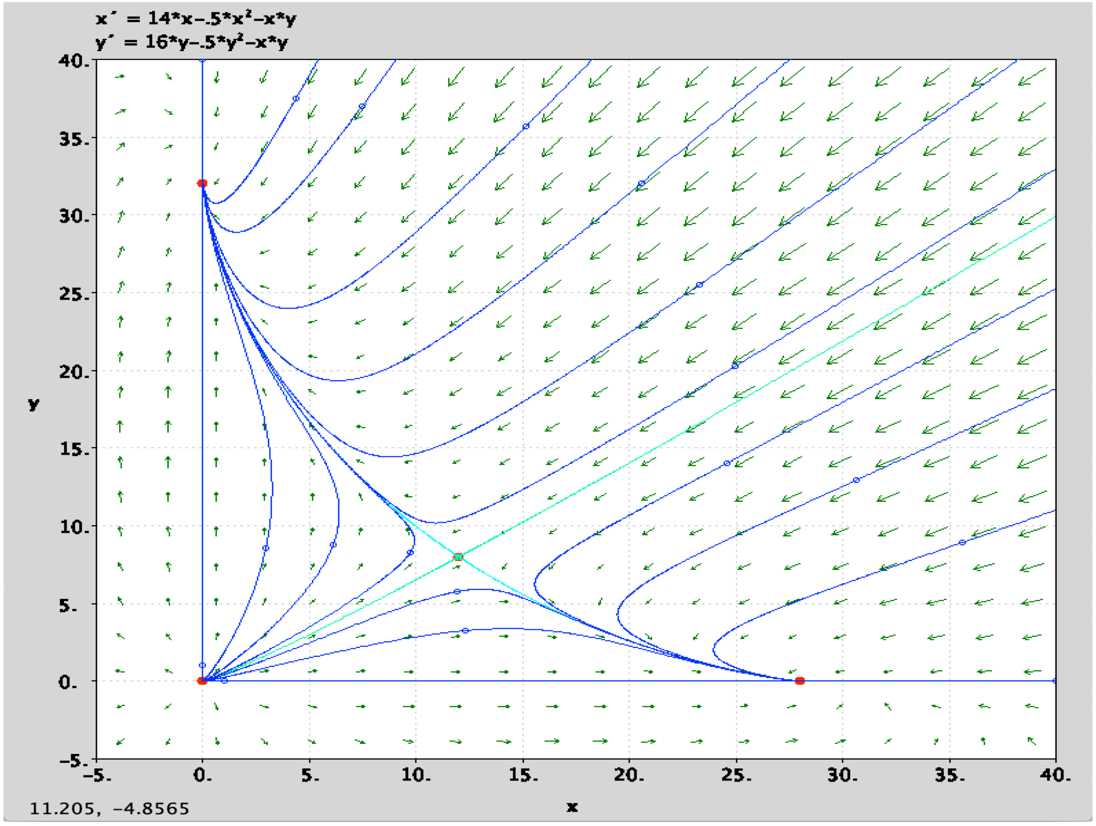
b) For the first order system in **w9.2** is the origin a stable or unstable equilibrium point? What is its precise classifcation based on the descriptions of isolated critical points in section 5.3?

w9.4 Here is a competition model which, unlike the example in the notes for Friday March 7 (that we won't get to until after break), there doesn't seem to be a happy two-population equilibrium solution.

a) Find the equilibrium solutions for this system of two competing logistic species algebraically. (You can look at the pplane picture below to verify your answers.)

$$x'(t) = 14\,x - .5\,x^2 - x\,y$$
$$y'(t) = 16\,y - .5\,y^2 - x\,y$$

b) Use the pplane picture below to classify each equilbrium solution as stable or unstable. Under magnification near each of these four equilibrium points the solution trajectories and phase portraits look like those for autonomous linear systems of differential equations such as we study in section 5.3. (Analytically this is because of linearization, which we'll be discussing). Classify each of the four equilibria based what these magnifications look like, choose one of the descriptors: saddle, nodal sink, nodal source, spiral sink, spiral source, or center for each of quilibrium solutions.



x' = 14*x-.5*x²-x*y
y' = 16*y-.5*y²-x*y
y
x
11.205, -4.8565